Analysis Test 1 (Solutions + Commentary) : 2016–17

1. A is bounded above (and non–empty, but that is implicit in the question). $\sup(A)$ is the least upper bound of the set A.

   Comments. Note that A does not have to be a finite set, or an interval of any kind. In particular its important to be aware that $\sup(A)$ *exists in many cases where A has no maximum element (see page 8 of notes).*

2. Given any pair of positive real numbers x and y, there exists a natural number N so that $Nx > y$.

   Comments. Many of you wrote that x and y were any real numbers. That isn't true, e.g. it fails if $x = 0$ and $y > 0$.

3. Since there exists an irrational number between any two rational numbers, we have
$$2.645730 < q < 2.645731 < \sqrt{7}.$$

   OR Use the fact that $\frac{1}{2}(2.64573 + \sqrt{7})$ is irrational (as the sum of a rational and an irrational is irrational, and dividing an irrational number by 2 cannot make it rational).

   Comments. $\sqrt{7}$ is irrational (by Theorem 1.2.1, as it is the square root of a prime number), so it cannot be equal to 2.645731*, which is rational. The point is that the first seven numbers in the (infinite) decimal expansion of $\sqrt{7}$ are* 2.645731*. Several of you tried to use a "theorem in the course" that there are an infinite number of irrational numbers between any two real numbers. That is true, but not written anywhere in the notes; so can't be used without an argument to justify it (it is, in fact, not difficult to deduce this from Theorem 1.2.2 and Theorem 1.4.5.)*

4. Given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that if $n > N$ then $|a_n - l| < \epsilon$.

   Comments. The precise wording is important, as is the order in which things appear; for example, N depends upon the choice of ϵ, so to put "for all $\epsilon > 0$" at the end of the definition, as some of you did, changes the meaning to a statement that is wrong.

5. Guess $\lim_{n\to\infty}\left(\frac{5}{7} - \frac{3}{2n^{1/3}}\right) = \frac{5}{7}$.

   Given $\epsilon > 0$, need to find $N \in \mathbb{N}$ so that $n > N$ implies $\frac{3}{2n^{1/3}} < \epsilon$.

Now $\frac{3}{2n^{1/3}} < \epsilon$ if and only if $n > 27/8\epsilon^3$. The Archimedean property tells us that there exists $N \in \mathbb{N}$ so that $N > 27/8\epsilon^3$. Then for any $n > N$ we have $n > 27/8\epsilon^3$, and the result follows.

Comments. Apart from the guess, this was done very badly, and that surprised me as the problem was almost identical to one that was done on the board at the beginning of Week 5 tutorials. For those who did make progress beyond the guess, there were some worrying algebraic errors; also misuse of inequalities – remember that if $a, b > 0$ and $a < b$ then $1/a > 1/b$.

Finally a few of you wrote $|5/7 - 3/2n^{\frac{1}{3}} - 5/7| = -3/2n^{\frac{1}{3}}$ which is negative! Please be clear that $|x|$ always picks out the "positive part" of your number, so if $x \geq 0, |x| = x$ and also, $|-x| = x$.

6. Since $-1 \leq \sin n \leq 1$, we have $0 \leq \frac{1-\sin(n)}{n^2} \leq \frac{2}{n^2} \to 0$ as $n \to \infty$.

   Then by the sandwich rule, the required limit exists and is zero.

   OR Use the fact that the sequence with nth term $\frac{1}{n^2}$ is null (i.e. it converges to zero), and that with nth term is $1 - \sin(n)$ is bounded. Then the product is a null sequence (this is proved in Problem 30).

   Comments. This question was done quite well by many of you. Please be very aware that in this course, you must explain how you get your results; so if you did the reasoning correctly, but omitted to say it was "by the sandwich rule", then you lost a mark.

   Many of you tried to use algebra of limits in a context where it isn't justified. You can only write $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n . \lim_{n\to\infty} b_n$ if you know that both (a_n) and (b_n) converge to a limit. But quite a few of you wrote $\lim_{n\to\infty} \frac{1-\sin(n)}{n^2} = \lim_{n\to\infty} 1/n^2 . \lim_{n\to\infty}(1 - \sin(n))$. But $1 - \sin(n)$ doesn't converge to a limit, it oscillates.

   Another common error that many of you made was to try to use the information given in the question and take "$k = 1 - \sin(n)$". But k is a real number, and $1 - \sin(n)$ is a function of n, so this makes no sense. In fact, by the same incorrect reasoning, you could take $k = n^2$ and then $\lim_{n\to\infty} k/n^2 = \lim_{n\to\infty} 1 = 1$!! Note that in my solution given above, using the sandwich rule, I took $k = 2$.