Analysis Test 2 (Solutions with Commentary) : 2016–17

1. (a) Given any sequence (x_n) with $x_n \in D_f \setminus \{a\}$ for all $n \in \mathbb{N}$, if $\lim_{n\to\infty} x_n = a$ then $\lim_{n\to\infty} f(x_n) = l$.
   (b) Given any $\epsilon > 0$ there exists $\delta > 0$ so that if $x \in D_f$ with $0 < |x - a| < \delta$ then $|f(x) - l| < \epsilon$.
   (c) $\lim_{x\to a} f(x)$ exists and equals $f(a)$.
   (d) $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists (and is finite).

This was just recalling important definitions (and one theorem) from notes/lectures. Common errors were to miss out the set $D_f \setminus \{a\}$ in (a), and to omit the word "exists" in (d) or to get the definition of differentiability completely wrong.

2. (a) $D_f = \mathbb{R} \setminus \{-6, 1, 6\}$.
   (b) For $x \in D_f$, $f(x) = \frac{1}{(6-x)(1-x)} = \frac{1}{(x-6)(x-1)}$.
   By algebra of limits, $\lim_{x\to -6} f(x) = \frac{1}{12.7} = \frac{1}{84}$.
   $\lim_{x\to 6} f(x)$ does not exist. To see this consider the sequence (x_n) where $x_n = 6 + 1/n$ for all $n \in \mathbb{N}$. Then

$$f(x_n) = \frac{1}{1/n(5 + 1/n)} = \frac{n}{5 + 1/n} \to \infty \text{ as } n \to \infty.$$

(a) was done well. (b) For the algebraic simplification, many of you didn't point out that you can only cancel $x+6$ when $x \neq -6$. A common careless error was to write $36 - x^2 = (x - 6)(x + 6)$, when it should be $(6 - x)(6 + x)$.

Many of you used sequences in discussions of both limits. One frequent problem (and this also came up in 3) and 4)) was students attempting to calculate $\lim_{x\to 6}$ using a special sequence such as $6 + 1/n$ which converges to 6. That isn't correct. To prove a limit of a function exists (see 1a)), you have to be able to do it for any sequence (in the domain) that converges to the point (which in this case is 6). On the other hand, to prove that a function doesn't have a limit at a point, its enough to show that divergence occurs for just one sequence. (Example 3.5 in the notes is the model here).

You didn't get very many marks if you used vague language about divergence, or started writing down quantities that involved dividing by zero (which is not allowed in analysis).

3. Left limit: $\lim_{x\uparrow 0} f(x) = 3$ and right limit: $\lim_{x\downarrow 0} f(x) = 1$. These are not equal, so $\lim_{x\to 0} f(x)$ does not exist. So f cannot be continuous at $x = 0$. By a theorem in the notes,[1] as f is not continuous at 0 it is not differentiable there.

   Alternatively, for the second part you can show that the right derivative $f'_+(x) = 10x$ for $x \geq 0$, so that $f'_+(0) = 0$, but the left derivative $f'_-(0)$ does not exist, i.e. the limit diverges to $-\infty$.

   Many of you did this very well. A common error was to obtain $f'_-(0) = -1$, *by erroneously putting* $f(0) = 3$ *in the limiting formula. But* $f(0) = 1$.

4. Left limit: $\lim_{x\uparrow 0} f(x) = \lim_{x\to 0}(1 - x) = 1$. Right limit: $\lim_{x\downarrow 0} f(x) = \lim_{x\to 0}(1+x) = 1$. These are equal to each other, and to $f(0) = 1$, and so f is continuous at $x = 0$.

   Left derivative: $\lim_{h<0,h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{1-h-1}{h} = -1$.

   Right derivative: $\lim_{h>0,h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{1+h-1}{h} = 1$.

   These are different, and so f is not differentiable at $x = 0$.

   Rather a lot of you seemed to think that the function is continuous at a point if the left limit equals the right limit. That isn't true. The property just stated is a necessary and sufficient condition for a function to have a limit at a point. For it to be continuous there, the limit must equal the value of the function. Compare Theorems 3.3.2 and 4.2.1 in the notes.

   Quite a few of you wrote down the values of left and right derivative without showing any working, because you formally differentiated. You can get away with this for the right derivative, but not for the left (think about it!).

[1] This is Theorem 5.2.1, but for tests and exams, you really shouldn't waste time memorising theorem numbers.