

MAS331 METRIC SPACES 2016-17

Solutions to Assignment 1

1. For $0 \leq x \leq \frac{\pi}{4}$, $\cos x \geq \sin x$ so $|f(x) - g(x)| = \cos x - \sin x$. But for $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$, $\sin x \geq \cos x$ so $|f(x) - g(x)| = \sin x - \cos x$. Hence

$$\begin{aligned} d_1(f, g) &= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx \\ &= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} \\ &= \frac{2}{\sqrt{2}} - 1 + (-1 + \frac{2}{\sqrt{2}}) \\ &= \frac{4}{\sqrt{2}} - 2 = 2(\sqrt{2} - 1). \end{aligned}$$

As $0 \leq \sin x, \cos x \leq 1$ for $0 \leq x \leq \frac{\pi}{2}$, it follows that $|f(x) - g(x)| \leq 1$ for $0 \leq x \leq \frac{\pi}{2}$. As $|f(0) - g(0)| = 1 = |f(\frac{\pi}{2}) - g(\frac{\pi}{2})|$, $d_\infty(f, g) = 1$.

(Alternatively, you can check that $\cos x - \sin x$ has no stationary points in $[0, \frac{\pi}{2}]$ so $|f(x) - g(x)|$ takes its maximum value at 0 or at $\frac{\pi}{2}$. $|f(x) - g(x)|$ decreases from 1 to 0 on $[0, \frac{\pi}{4}]$ and increases from 0 to 1 on $[\frac{\pi}{4}, \frac{\pi}{2}]$.)

2. (M1) $d(\mathbf{x}, \mathbf{y})$ is a finite sum of non-negative numbers and so is itself non-negative.

$$d(\mathbf{x}, \mathbf{x}) = \sum_{i=1}^n a_i |x_i - x_i| = 0$$

Conversely if $d(\mathbf{x}, \mathbf{y}) = 0$ then $\sum_{i=1}^n a_i |x_i - y_i| = 0$ and so $a_i |x_i - y_i| = 0$ for $1 \leq i \leq n$. But $a_i > 0$, so $|x_i - y_i| = 0$, i.e. $x_i = y_i$ ($1 \leq i \leq n$). Hence $\mathbf{x} = \mathbf{y}$.

$$(M2) \quad d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n a_i |x_i - y_i| = \sum_{i=1}^n a_i |y_i - x_i| = d(\mathbf{y}, \mathbf{x}).$$

(M3)

$$\begin{aligned}d(\mathbf{x}, \mathbf{z}) &= \sum_{i=1}^n a_i |x_i - z_i| \\&= \sum_{i=1}^n a_i |x_i - y_i + y_i - z_i| \\&\leq \sum_{i=1}^n a_i (|x_i - y_i| + |y_i - z_i|) \\&= \sum_{i=1}^n a_i |x_i - y_i| + \sum_{i=1}^n a_i |y_i - z_i| \\&= d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}).\end{aligned}$$

3. Let $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$.

(b)

$$\begin{aligned}d_1(a, b) &= \sum_{1 \leq i \leq n} |a_i - b_i| \\&\leq n \max_{1 \leq i \leq n} |a_i - b_i|\end{aligned}$$

because each of the n terms in the sum is \leq their maximum. Thus $d_1(a, b) \leq n d_\infty(a, b)$.

(c) For $1 \leq i \leq n$, $(a_i - b_i)^2 = |a_i - b_i|^2 \leq (\max_{1 \leq i \leq n} |a_i - b_i|)^2$ so

$$\begin{aligned}d_2(a, b) &= \sqrt{\sum_{1 \leq i \leq n} (a_i - b_i)^2} \\&\leq \sqrt{n (\max_{1 \leq i \leq n} |a_i - b_i|)^2} \\&= \sqrt{n} d_\infty(a, b).\end{aligned}$$

(e) From the proof of (b), $d_1(a, b) = n d_\infty(a, b)$ if and only if all n of the numbers $|a_i - b_i|$ are equal (so that each is equal to the maximum and their sum is n times the maximum). For example it happens for $(1, 2, 3)$ and $(2, 3, 4)$ or $(2, 1, 4)$.

From the proof of (c), $d_2(a, b) = \sqrt{n} d_\infty(a, b)$ if and only if all n of the numbers $|a_i - b_i|$ are equal (so that each is equal to the maximum and the sum inside the square root is n times the maximum). This is the same condition as for $d_1(a, b) = n d_\infty(a, b)$.

4. By (M3) $d(x, z) \leq d(x, y) + d(y, z)$ and so

$$d(x, z) - d(y, z) \leq d(x, y) \dots \text{(i)}$$

Since this holds for all $x, y, z \in X$ we may interchange x and y in (i) to get

$$d(y, z) - d(x, z) \leq d(y, x) = d(x, y) \dots \text{(ii)}$$

by (M2). Combining (i) and (ii) together gives the desired result:

$$|d(x, z) - d(y, z)| \leq d(x, y) \dots \text{(iii)}$$

For the second result we have

$$\begin{aligned} |d(x, y) - d(a, b)| &= |d(x, y) - d(y, a) + d(y, a) - d(a, b)| \\ &\leq |d(x, y) - d(y, a)| + |d(y, a) - d(a, b)| \\ &= |d(x, y) - d(a, y)| + |d(y, a) - d(b, a)| \\ &\leq d(x, a) + d(y, b), \end{aligned}$$

where we used (iii) twice to get the last line.