MAS331 METRIC SPACES

Solutions to Assignment 1

1. For $0 \le x \le \frac{\pi}{4}$, $\cos x \ge \sin x$ so $|f(x) - g(x)| = \cos x - \sin x$. But for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$, $\sin x \ge \cos x$ so $|f(x) - g(x)| = \sin x - \cos x$. Hence

$$\begin{aligned}
& d_1(f,g) \\
= \quad & \int_0^{\pi/4} (\cos x - \sin x)\ dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\ dx \\
= \quad & [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} \\
= \quad & \frac{2}{\sqrt{2}} - 1 + (-1 + \frac{2}{\sqrt{2}}) \\
= \quad & \frac{4}{\sqrt{2}} - 2 = 2(\sqrt{2} - 1).
\end{aligned}$$

As $0 \le \sin x, \cos x \le 1$ for $0 \le x \le \frac{\pi}{2}$, it follows that $|f(x) - g(x)| \le 1$ for $0 \le x \le \frac{\pi}{2}$. As $|f(0) - g(0)| = 1 = |f(\frac{\pi}{2}) - g(\frac{\pi}{2})|$, $d_\infty(f,g) = 1$.

(Alternatively, you can check that $\cos x - \sin x$ has no stationary points in $[0, \frac{\pi}{2}]$ so $|f(x) - g(x)|$ takes its maximum value at 0 or at $\frac{\pi}{2}$. $|f(x) - g(x)|$ decreases from 1 to 0 on $[0, \frac{\pi}{4}]$ and increases from 0 to 1 on $[\frac{\pi}{4}, \frac{\pi}{2}]$.)

2. (M1) $d(\mathbf{x}, \mathbf{y})$ is a finite sum of non-negative numbers and so is itself non-negative.

$$d(\mathbf{x}, \mathbf{x}) = \sum_{i=1}^{n} a_i |x_i - x_i| = 0$$

Conversely if $d(\mathbf{x}, \mathbf{y}) = 0$ then $\sum_{i=1}^{n} a_i |x_i - y_i| = 0$ and so $a_i |x_i - y_i| = 0$ for $1 \le i \le n$. But $a_i > 0$, so $|x_i - y_i| = 0$, i.e. $x_i = y_i$ $(1 \le i \le n)$. Hence $\mathbf{x} = \mathbf{y}$.

(M2) $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} a_i |x_i - y_i| = \sum_{i=1}^{n} a_i |y_i - x_i| = d(\mathbf{y}, \mathbf{x})$.

(M3)

$$\begin{aligned} d(\mathbf{x},\mathbf{z}) &= \sum_{i=1}^{n} a_i|x_i - z_i| \\ &= \sum_{i=1}^{n} a_i|x_i - y_i + y_i - z_i| \\ &\leq \sum_{i=1}^{n} a_i(|x_i - y_i| + |y_i - z_i|) \\ &= \sum_{i=1}^{n} a_i|x_i - y_i| + \sum_{i=1}^{n} a_i|y_i - z_i|) \\ &= d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}). \end{aligned}$$

3. Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n$.

(b)

$$\begin{aligned} d_1(a,b) &= \sum_{1\leq i\leq n} |a_i - b_i| \\ &\leq n \max_{1\leq i\leq n} |a_i - b_i| \end{aligned}$$

because each of the n terms in the sum is $\leq$ their maximum. Thus $d_1(a,b) \leq nd_\infty(a,b)$.

(c) For $1 \leq i \leq n$, $(a_i - b_i)^2 = |a_i - b_i|^2 \leq (\max_{1\leq i\leq n} |a_i - b_i|)^2$ so

$$\begin{aligned} d_2(a,b) &= \sqrt{\sum_{1\leq i\leq n} (a_i - b_i)^2} \\ &\leq \sqrt{n(\max_{1\leq i\leq n} |a_i - b_i|)^2} \\ &= \sqrt{n} d_\infty(a,b). \end{aligned}$$

(e) From the proof of (b), $d_1(a,b) = nd_\infty(a,b)$ if and only if all n of the numbers $|a_i - b_i|$ are equal (so that each is equal to the maximum and their sum is n times the maximum). For example it happens for $(1,2,3)$ and $(2,3,4)$ or $(2,1,4)$.

From the proof of (c), $d_2(a,b) = \sqrt{n} d_\infty(a,b)$ if and only if all n of the numbers $|a_i - b_i|$ are equal (so that each is equal to the maximum and the sum inside the square root is n times the maximum). This is the same condition as for $d_1(a,b) = nd_\infty(a,b)$.

4. By (M3) $d(x,z) \le d(x,y) + d(y,z)$ and so

$$d(x,z) - d(y,z) \le d(x,y) \ldots \text{(i)}$$

Since this holds for all $x, y, z \in X$ we may interchange x and y in (i) to get

$$d(y,z) - d(x,z) \le d(y,x) = d(x,y) \ldots \text{(ii)}$$

by (M2). Combining (i) and (ii) together gives the desired result:

$$|d(x,z) - d(y,z)| \le d(x,y) \ldots \text{(iii)}$$

For the second result we have

$$\begin{aligned}
|d(x,y) - d(a,b)| &= |d(x,y) - d(y,a) + d(y,a) - d(a,b)| \\
&\le |d(x,y) - d(y,a)| + |d(y,a) - d(a,b)| \\
&= |d(x,y) - d(a,y)| + |d(y,a) - d(b,a)| \\
&\le d(x,a) + d(y,b),
\end{aligned}$$

where we used (iii) twice to get the last line.