MAS331 METRIC SPACES 2016-17

Solutions to Assignment 2

1. $|f_n(x) - g(x)| = |\frac{\cos(nx)}{n}| \leqslant \frac{1}{n}$ so $d_\infty(f_n, g) \leqslant \frac{1}{n}$. Also $|f_n(0) - g(0)| = \frac{1}{n}$ so $d_\infty(f_n, g) = \frac{1}{n} \to 0$. Hence $(f_n) \to g$ in $(C[0, 2\pi], d_\infty)$.

   By Proposition 2.15, (f_n) converges to g in $(C[0, 2\pi], d_1)$ and (f_n) converges pointwise to g in $C[0, 2\pi]$.

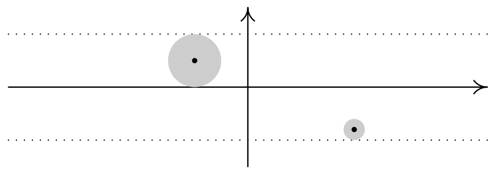
2. In the set $C[0, 1]$, let $F = \{f \in C[0, 1] : f(1) = 0\}$.

   (a) Let (f_n) be a sequence of functions in F that converges to some limit f in $(C[0, 1], d_\infty)$. We have to show that $f \in F$ i.e. $f(1) = 0$. By Proposition 2.15, (f_n) converges pointwise to f in $C[0, 1]$. In particular $(f_n(1))$ converges to $f(0)$. But each $f_n \in F$ so $(f_n(1))$ is the constant sequence (0) which converges to 0. Hence $f(1) = 0$. Therefore F is a closed subset of $(C[0, 1], d_\infty)$.

   (b) $d_1(f_n, g) = \int_0^1 (|1 - x^n - 1| dx = \int_0^1 x^n \, dx = \frac{1}{n+1} < \frac{1}{n}$. As $\frac{1}{n} \to 0$, it follows from the Sandwich Rule that $d_1(f_n, g) \to 0$. Hence $(f_n) \to g$ in $(C[0, 1], d_1)$.

   For each n, $f_n(1) = 0$ so $f_n \in F$. However $g(1) = 1$ so $g \notin F$. As $(f_n) \to g$ in $(C[0, 1], d_1)$, it follows that F is not a closed subset of $(C[0, 1], d_1)$.

3. It helps to first draw a diagram. This one is for A_3 and shows appropriate open balls for two points of A_3.



   (a) To see that A_1 is not closed, consider the sequence $((0, -1 + \frac{1}{n}))$ which has limit $(0, -1)$. All terms in the sequence are in A_1 but the limit is not.

   To see that A_1 is not open, take the point $(0, 1)$ which is in A_1. But any open ball $B((0, 1), r)$ contains the point $(0, 1 + r/2)$ which is not in A_1.

   Thus A_1 is neither open nor closed in $\mathbb{R}^2$.

(b) Let (x_n, y_n) be a sequence of points tending to a limit $(x, y) \in A_2$. Then $y_n \to y$. Each $y_n \in [-1, 1]$ which is closed in $\mathbb{R}$ so $y \in [-1, 1]$, that is $-1 \leq y \leq 1$. Hence $(x, y) \in A_2$ and so A_2 is closed.

(c) *Method 1.* Let $(x, y) \in A_3$. Then $-1 < y < 1$ so $y + 1 > 0$ and $1 - y > 0$. Let $r = \min(1 - y, y + 1)$. (In the diagram, the larger open ball has radius $1 - y$ and the smaller one has radius $y - (-1) = y + 1$.) Thus $1 - y \geqslant r$ and $y + 1 \geqslant r$. Let $(a, b) \in B((x, y), r)$. We have to show that $-1 < b < 1$ so that $(a, b) \in A_3$ and $B((x, y), r) \subseteq A_3$. Now $|y - b| \leq d_2((x, y), (a, b)) < r$ so $-r < b - y < r$ and $y - r < b < y + r$. As $1 - y \geqslant r$, we have $1 \geqslant r + y > b$ and as $y + 1 \geqslant r$, we have $b > y - r \geqslant -1$ so $-1 < b < 1$, $(a, b) \in A_3$ and $B((x, y), r) \subseteq A_3$. Thus A_3 is open.

Method 2. Show that the complement $\mathbb{R}^2 \backslash A_3$ is closed and then apply Theorem 3.15. This complement is the union of the sets $\{(x, y) \in \mathbb{R}^2 : y \geqslant 1\}$ and $\{(x, y) \in \mathbb{R}^2 : y \leqslant -1\}$ and it is enough, by Proposition 3.18, to show that each of these is closed, and you can do that by a similar argument to (b).

Method 3. Use Theorem 4.13. First use Proposition 2.9 and the sequential definition of continuity to show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y$ is continuous. Then note that $A_3 = f^{-1}((-1, 1))$, which is open, since $(-1, 1)$ is open in $\mathbb{R}$. This method can also be used as an alternative approach to (b).

4. (i) Suppose that $a_n \to x$ under d_2. Let $\varepsilon > 0$. Then there exists N such that $a_n \in B_2(x, \varepsilon) \subseteq B_\infty(a, \varepsilon)$ whenever $n > N$. Hence $a_n \in B_\infty(x, \varepsilon)$ whenever $n > N$ and so $a_n \to x$ under d_∞.

Conversely, suppose that $a_n \to x$ under d_∞. Let $\varepsilon > 0$. Then $\varepsilon/\sqrt{m} > 0$ so there exists N such that $a_n \in B_\infty(x, \varepsilon/\sqrt{m}) \subseteq B_2(a, \varepsilon)$ whenever $n > N$. Hence $a_n \in B_2(x, \varepsilon)$ whenever $n > N$ and so $a_n \to x$ under d_2.

Thus $a_n \to x$ under d_2 if and only if $a_n \to x$ under d_∞.

(ii) Suppose that A is closed under d_2 and let (a_n) be a sequence in A converging, under d_∞, to $x \in X$. Then, by (i), $(a_n) \to x$ under d_2. As A is closed under d_2, $x \in A$ so A is closed under d_∞.

Conversely suppose that A is closed under d_∞ and let (a_n) be a sequence in A converging, under d_2, to $x \in X$. Then, by (i), $(a_n) \to x$ under d_∞. As A is closed under d_∞, $x \in A$ so A is closed under d_2.

(iii) Suppose that A is open under d_∞. There exists $r > 0$ such that $B_\infty(a, r) \subseteq A$. But, $B_2(a, r) \subseteq B_\infty(a, r)$ so $B_2(a, r) \subseteq A$ and therefore A is open under d_2.

Conversely suppose that A is open under d_2. There exists $r > 0$ such that $B_2(a, r) \subseteq A$. But $B_\infty(a, r/\sqrt{m}) \subseteq B_2(a, r)$ so $B_\infty(a, r/\sqrt{m}) \subseteq A$ and therefore A is open under d_∞.