MAS331 METRIC SPACES

Solutions to Assignment 2

1. Since $|\cos(y)| \leq 1$ for all $y \in \mathbb{R}$,

$$|f_n(x) - g(x)| = |\frac{\cos(nx)}{n}| \leqslant \frac{1}{n},$$

so $d_\infty(f_n, g) \leqslant \frac{1}{n}$. Also $|f_n(0) - g(0)| = \frac{1}{n}$ so $d_\infty(f_n, g) = \frac{1}{n} \to 0$. Hence $(f_n) \to g$ in $(C[0, 2\pi], d_\infty)$.

By Proposition 2.15, (f_n) converges to g in $(C[0, 2\pi], d_1)$ and (f_n) converges pointwise to g in $C[0, 2\pi]$.

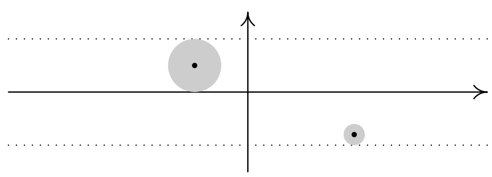
2. In the set $C[0, 1]$, let $F = \{f \in C[0, 1] : f(1) = 0\}$.

(a) Let (f_n) be a sequence of functions in F that converges to some limit f in $(C[0, 1], d_\infty)$. We have to show that $f \in F$ i.e. $f(1) = 0$. By Proposition 2.15, (f_n) converges pointwise to f in $C[0, 1]$. In particular $(f_n(1))$ converges to $f(0)$. But each $f_n \in F$ so $(f_n(1))$ is the constant sequence (0) which converges to 0. Hence $f(1) = 0$. Therefore F is a closed subset of $(C[0, 1], d_\infty)$.

(b) $d_1(f_n, g) = \int_0^1 (|1 - x^n - 1| dx = \int_0^1 x^n \, dx = \frac{1}{n+1} < \frac{1}{n}$. As $\frac{1}{n} \to 0$, it follows from the Sandwich Rule that $d_1(f_n, g) \to 0$. Hence $(f_n) \to g$ in $(C[0, 1], d_1)$.

For each n, $f_n(1) = 0$ so $f_n \in F$. However $g(1) = 1$ so $g \notin F$. As $(f_n) \to g$ in $(C[0, 1], d_1)$, it follows that F is not a closed subset of $(C[0, 1], d_1)$.

3. It helps to first draw a diagram. This one is for A_3 and shows appropriate open balls for two points of A_3.



(a) To see that A_1 is not closed, consider the sequence $((0, -1 + \frac{1}{n}))$ which has limit $(0, -1)$. All terms in the sequence are in A_1 but the limit is not.

To see that A_1 is not open, take the point $(0,1)$ which is in A_1. But any open ball $B((0,1),r)$ contains the point $(0,1+r/2)$ which is not in A_1.

Thus A_1 is neither open nor closed in $\mathbb{R}^2$.

(b) Let (x_n, y_n) be a sequence of points in A_2 tending to a limit $(x,y) \in \mathbb{R}^2$. Then $y_n \to y$. Each $y_n \in [-1,1]$ which is closed in $\mathbb{R}$ so $y \in [-1,1]$, that is $-1 \leq y \leq 1$. Hence $(x,y) \in A_2$ and so A_2 is closed.

(c) *Method 1.* Let $(x,y) \in A_3$. Then $-1 < y < 1$ so $y+1 > 0$ and $1-y > 0$. Let $r = \min(1-y, y+1)$. (In the diagram, the larger open ball has radius $1-y$ and the smaller one has radius $y-(-1) = y+1$.) Thus $1-y \geqslant r$ and $y+1 \geqslant r$. Let $(a,b) \in B((x,y),r)$. We have to show that $-1 < b < 1$ so that $(a,b) \in A_3$ and $B((x,y),r) \subseteq A_3$. Now $|y-b| \leq d_2((x,y),(a,b)) < r$ so $-r < b-y < r$ and $y-r < b < y+r$. As $1-y \geqslant r$, we have $1 \geqslant r+y > b$ and as $y+1 \geqslant r$, we have $b > y-r \geqslant -1$ so $-1 < b < 1$, $(a,b) \in A_3$ and $B((x,y),r) \subseteq A_3$. Thus A_3 is open.

Method 2. Show that the complement $\mathbb{R}^2 \backslash A_3$ is closed and then apply Theorem 3.15. This complement is the union of the sets $\{(x,y) \in \mathbb{R}^2 : y \geqslant 1\}$ and $\{(x,y) \in \mathbb{R}^2 : y \leqslant -1\}$ and it is enough, by Proposition 3.18, to show that each of these is closed, and you can do that by a similar argument to (b).

Method 3. Use Theorem 4.13. First use Proposition 2.9 and the sequential definition of continuity to show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = y$ is continuous. Then note that $A_3 = f^{-1}((-1,1))$, which is open, since $(-1,1)$ is open in $\mathbb{R}$. This method can also be used as an alternative approach to (b).

4. (i) Suppose that $a_n \to x$ under d_2. Let $\varepsilon > 0$. Then there exists N such that $a_n \in B_2(x,\varepsilon) \subseteq B_\infty(a,\varepsilon)$ whenever $n > N$. Hence $a_n \in B_\infty(x,\varepsilon)$ whenever $n > N$ and so $a_n \to x$ under d_∞.

Conversely, suppose that $a_n \to x$ under d_∞. Let $\varepsilon > 0$. Then $\varepsilon/\sqrt{m} > 0$ so there exists N such that $a_n \in B_\infty(x, \varepsilon/\sqrt{m}) \subseteq B_2(a,\varepsilon)$ whenever $n > N$. Hence $a_n \in B_2(x,\varepsilon)$ whenever $n > N$ and so $a_n \to x$ under d_2.

Thus $a_n \to x$ under d_2 if and only if $a_n \to x$ under d_∞.

(ii) Suppose that A is closed under d_2 and let (a_n) be a sequence in A converging, under d_∞, to $x \in X$. Then, by (i), $(a_n) \to x$ under d_2. As A is closed under d_2, $x \in A$ so A is closed under d_∞.

Conversely suppose that A is closed under d_∞ and let (a_n) be a sequence in A converging, under d_2, to $x \in X$. Then, by (i), $(a_n) \to x$ under d_∞. As A is closed under d_∞, $x \in A$ so A is closed under d_2.

(iii) Suppose that A is open under d_∞. There exists $r > 0$ such that $B_\infty(a, r) \subseteq A$. But, $B_2(a, r) \subseteq B_\infty(a, r)$ so $B_2(a, r) \subseteq A$ and therefore A is open under d_2.

Conversely suppose that A is open under d_2. There exists $r > 0$ such that $B_2(a, r) \subseteq A$. But $B_\infty(a, r/\sqrt{m}) \subseteq B_2(a, r)$ so $B_\infty(a, r/\sqrt{m}) \subseteq A$ and therefore A is open under d_∞.

Feedback In 1) a lot of you wrote $|\cos(nx)| \leq 1$ so $\sup|\cos(nx)| = 1$. That is not correct reasoning. In fact we also have $|\cos(nx)| \leq 100$ but $\sup|\cos(nx)| \neq 100$. In this case, to show $\sup|\cos(nx)| = 1$, you must find a value of x at which the sup is attained, and $x = 0$ will do (there are others).

2) and 3) were very well done overall.

4) Some students were not using the correct definition of convergence. It is not an acceptable answer to write something like "Suppose that $a_n \to x$ under d_2. Let $\epsilon > 0$. Then $a_n \in B_2(x, \epsilon) \subseteq B_\infty(a, \epsilon)$. Hence $a_n \in B_\infty(x, \epsilon)$ and so $a_n \to x$ under d_∞." You need to start with "given $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n > N, \ldots$."