MAS331/6352 METRIC SPACES

Solutions to Assignment 3

1. We use the Differential Criterion. Let $f(x) = \cos(\sin(x))$. Then, for all $x \in \mathbb{R}$, $|f'(x)| = |-\sin(\sin(x))\cos(x)| \leq |\sin(\sin(x))| \leq \sin(1)$ as $-1 \leq \sin(x) \leq 1$, and $x \to \sin(x)$ is monotonic increasing on the interval $[-1, 1]$. Thus f satisfies the Differential Criterion with $k = \sin(1)$, and so it is a contraction.

   As $\mathbb{R}$ is complete, the Contraction Mapping Principle tells us that there is a unique element $x \in \mathbb{R}$ such that $\cos(\sin(x)) = x$.

2. Take $x, y \in X$. Now since f is a contraction, for any $u, v \in X$ we have $d(f(u), f(v)) \leqslant kd(u, v)$. In this formula take $u = g(x)$, $v = g(y)$ to get

   $$d(f \circ g(x), f \circ g(y)) = d(f(g(x)), f(g(y))) \leqslant kd(g(x), g(y)) \leqslant kk'd(x, y)$$

   where the last step comes because g is also a contraction. Since $0 \leqslant kk' < 1$, $f \circ g$ is a contraction.

   If x is a fixed point of $f \circ g$ then $f(g(x)) = f \circ g(x) = x$. Applying g to this equation gives $g(f(g(x))) = g(x)$, or in other words $g \circ f(g(x)) = g(x)$. Thus $g(x)$ is a fixed point of $g \circ f$.

3. $a = \frac{5}{9}$ and $b = -\frac{160}{9}$ so that $f(32) = 0$ and $f(212) = 100$. For $x, y \in \mathbb{R}$,

   $$|f(x) - f(y)| = \left|\frac{5}{9}x - \frac{160}{9} - \left(\frac{5}{9}y - \frac{160}{9}\right)\right| = \frac{5}{9}|x - y|.$$

   As $\frac{5}{9} < 1$, f is a contraction.

   As $\mathbb{R}$ is complete, f has a unique fixed point but, in this case, that is easily confirmed algebraically. The equation $x = \frac{5}{9}x - \frac{160}{9}$ has a unique solution, $x = -40$.

4. Let (f_n) be the sequence of functions in $C[0, 1]$ defined by $f_n(x) = x^n$, for all $x \in [0, 1], n \in \mathbb{N}$. Then we have seen in Example 2.16 that $(f_n) \to f$ in $(C[0, 1], d_1)$, where $f(x) = 0$ for all $x \in [0, 1]$ but $(f_n) \not\to f$ in $(C[0, 1], d_\infty)$. But $\theta(f_n) = f_n$ so $(f_n) \to f$ in $(C[0, 1], d_1)$ but $(\theta(f_n)) \not\to f$ in $(C[0, 1], d_\infty)$. Therefore, as a function from $(C[0, 1], d_1)$ to $(C[0, 1], d_\infty)$, θ is not continuous.

   Now let (f_n) be any sequence of functions in $C[0, 1]$ converging, in $(C[0, 1], d_\infty)$, to f, for some $f \in C[0, 1]$. By Proposition 2.15, $(f_n) \to f$ in $(C[0, 1], d_1)$, that is, $(\theta(f_n)) \to f$ in $(C[0, 1], d_1)$. Therefore, as a function from $(C[0, 1], d_\infty)$ to $(C[0, 1], d_1)$, θ is continuous.

5. Looking at the picture or the formula, it's clear that $f_n(\frac{1}{n}) = 1$, while $f_{2n}(\frac{1}{n}) = 0$, so that

$$f_n(\tfrac{1}{n}) - f_{2n}(\tfrac{1}{n}) = 1.$$

   Hence

$$d_\infty(f_{2n}, f_n) = \sup\{|f_{2n}(x) - f_n(x)| \, : \, x \in [0,1]\} \geqslant 1.$$

   This means that (f_n) cannot be Cauchy. For, if it were, then there would be some N such that $d_\infty(f_n, f_m) < 1$ for all $n, m > N$. But if we pick any $n > N$, and set $m = 2n$, then we know $d_\infty(f_n, f_m) \geqslant 1$, which is a contradiction.

   If $m \geqslant 2n$ then $\frac{1}{n} \geqslant \frac{2}{m}$ so $f_m(\frac{1}{n}) = 0$ and, as above for $2n$,

$$d_\infty(f_m, f_n) = \sup\{|f_m(x) - f_n(x)| \, : \, x \in [0,1]\} \geqslant 1.$$

   If a subsequence (f_{n_k}) is a Cauchy sequence then there exists K such that $d_\infty(f_{n_k}, f_{n_\ell}) < 1$ for all $k, \ell > K$. But if $k > K$ there exists $\ell > K$ such that $n_\ell \geq 2n_k$ so $d_\infty(f_{n_k}, f_{n_\ell}) \geqslant 1$, by the above argument. So (f_n) has no Cauchy subsequence.