Review of D.Bakry, I.Gentil, M.Ledoux, "Analysis and Geometry of Markov Diffusion Operators", Grundlehren der mathematischen Wissenschaften **348**, Springer International Publishing Switzerland 2014, ISBN 978-3-319-00226-2, 552 pages

This impressive monograph is about an important, and highly active, area that straddles the fertile land occupied by both probability and analysis. Although a broad range of topics are dealt with, the central theme is the role of the Γ calculus (which we describe below) as a unifying context for a number of important functional inequalities.

We begin with some essential background. Much of modern probabilistic modeling in continuous time is concerned with (time-homogeneous) Markov processes $(X(t), t \geq 0)$, taking values in some state space M; so dependence of the future of the process on the past is only through the present. Each such process gives rise, through averaging, to a family of linear operators $(P_t, t \geq 0)$ on the space of bounded measurable functions on M. To be precise, for each such function f and $x \in M$,

$$(P_t f)(x) = \mathbb{E}_x(f(X(t))),$$

where $\mathbb{E}_x$ is mathematical expectation (i.e. a Lebesgue integral with respect to the underlying probability measure), and the subscript x indicates conditioning on the starting value $X(0) = x$. These operators form a *semigroup* in that $P_{s+t} = P_s P_t$ for all $s, t \geq 0$. In this book, it is always assumed that there is an invariant measure μ so that

$$\int_M P_t f(x) d\mu(x) = \int_M f(x) d\mu(x).$$

Then for many situations of interest, $(P_t, t \geq 0)$ is a strongly continuous, self-adjoint contraction semigroup on the Hilbert space $L^2(M, \mu)$, in which case it has an *infinitesimal generator* L so that $P_t = e^{tL}$ (in the spectral theory sense). Another vital tool is the *Dirichlet form* $\mathcal{E}(f) = -\langle Lf, f \rangle$, where the angle brackets denote the Hilbert space inner product. For example, if $(X(t), t \geq 0)$ is Brownian motion on $M = \mathbb{R}^d$, then $(P_t, t \geq 0)$ is the heat semigroup, $L = \frac{1}{2}\Delta$, where Δ is the Laplacian, μ is Lebesgue measure, and $\mathcal{E}(f) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx$ is the energy integral. Much of the theory in the book is concerned with more general second order diffusion operators, so if $M = \mathbb{R}^d$

$$(Lf)(x) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j f(x) + \sum_{i=1}^{d} b_i(x)\partial_i f(x).$$

Returning to the general set-up, Γ calculus is concerned with the *carré du champ operator*

$$\Gamma(f,g) = \frac{1}{2}[L(fg) - fL(g) - L(f)g],$$

and its iterate:

$$\Gamma_2(f,g) = \frac{1}{2}[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(Lf,g)].$$

We say that a *curvature-dimension inequality* $\mathrm{CD}(\rho,n)$ holds for $\rho \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{\infty\}$ if

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f) + \frac{1}{n}(Lf)^2,$$

on some suitable set of functions. The name for this inequality is motivated by the fact that, when the process is Brownian motion on a Riemannian manifold, the optimal choice for n is the dimension of the manifold, and that for ρ is the lowest eigenvalue of the Ricci curvature tensor.

The main part of the book is concerned with three very important inequalities, and their variants, these being:

The Poincaré Inequality:

$$\mathrm{V}ar_\mu(f) \leq C_1\mathcal{E}(f),$$

where the variance $\mathrm{V}ar_\mu(f) = \int_M f^2 d\mu - \left(\int_M f d\mu\right)^2$.

The Logarithmic Sobolev Inequality:

$$\mathrm{Ent}_\mu(f^2) \leq C_2\mathcal{E}(f) + C_3\int_M f^2 d\mu,$$

where the entropy $\mathrm{Ent}_\mu(f) = \int_M f\log(f)d\mu - \left(\int_M f d\mu\right)\log\left(\int_M f d\mu\right)$.

The Sobolev Inequality:

$$||f||_p^2 \leq C_4||f||_2^2 + C_5\mathcal{E}(f),$$

where the p-norm $||f||_p = \left(\int_M |f|^p d\mu\right)^{\frac{1}{p}}$ for $p \geq 2$, and $C_1, \ldots, C_5$ are non-negative constants.

Each of these inequalities is implied by a suitable form of $\mathrm{CD}(\rho,n)$, and each has a range of important applications. When the Poincaré inequality holds, we have L^2- exponential convergence of the semigroup to equilibrium, and the generator has a *spectral gap* (which is important in applications to physics); the logarithmic-Sobolev inequality is equivalent to *hypercontractivity* of the semigroup, i.e. there exists $1 < q(t) < \infty$ so that

$$||P_t f||_{q(t)} \leq e^{M(t)} ||f||_p,$$

for $M(t) \geq 0$, and this property is useful in statistical mechanics; Sobolev inequalities have a wide range of applications, for example, to heat kernel bounds; moreover, the Sobolev inequality implies that the semigroup is *ultracontractive*, hence it is compact and has a discrete spectrum.

The last part of the book deals with various generalisations of these inequalities, as well as links with the potential-theoretic notion of capacity, and to transportation inequalities. The book closes with a number of useful appendices providing background in functional analysis, stochastic calculus and differential geometry. Finally, on page 521, the authors write, quite unfairly in the reviewer's opinion, that "After all, the content of this book is nothing but a sequence of recipes," and what follows is a genuine recipe for Chicken "Gaston Gérard". This looks very tasty, but the reviewer hasn't tried it yet.

I am very much aware that this brief review has failed to do justice to the rich selection of topics in this book. It is written with great clarity and style, and was clearly a labour of love for the authors. I am convinced that it will be a valuable resource for researchers in analysis and probability for many years to come.

David Applebaum, School of Mathematics and Statistics, University of Sheffield.