Martingale transform and Lévy Processes on Lie Groups

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Talk at *"Nonlocal Operators: Analysis, Probability, Geometry and Applications"*,

ZiF, Bielefeld.

9- 14 July 2012

Talk based on joint work with Rodrigo Bañuelos (Purdue)

http://arxiv.org/abs/1206.1560

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Burkholder Inequality

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We will denote by $[M] := [M, M]$ the quadratic co-variation process of M.

$$[M]_t := \text{l.i.p}_{|\mathcal{P}| \to 0} \sum_{i=1}^{N-1} (M_{t \wedge t_{i+1}} - M_{t \wedge t_i})^2,$$

where $\mathcal{P} = \{0 = t_0 < \ldots < t_N < \infty\}$ and $|\mathcal{P}| := \max_{1 \leq i \leq N} |t_i - t_{i-1}|$.

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Definition. We say that N is *differentially subordinate* to M if $|N_0| \leq |M_0|$ and $([M]_t - [N]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t.

Theorem (Generalised Burkholder Inequality)

(*Burkholder, Bañuelos-Wang, Wang*)

If N is differentially subordinate to M, then

$$\|N_T\|_p \leq (p^* - 1)\|M_T\|_p, \tag{1.1}$$

for all $1 < p < \infty$ *and all* $T > 0$ *where the sharp constant*

$$p^* := \max\left\{p, q : \frac{1}{p} + \frac{1}{q} = 1\right\}.$$

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- $\phi(0) = e$ (a.s.),
- ϕ has stationary and independent right increments, where the right increment between s and a later time t is the random variable $\phi(s)^{-1}\phi(t)$,
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Law of $\phi(t)$ is $p_t(A) = P(\phi(t) \in A)$.

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- A measure ν defined on $\mathcal{B}(G)$ is called a *Lévy measure* whenever $\nu(\{e\}) = 0$,

$$\int_U \left(\sum_{i=1}^{n} x_i(\tau)^2 \right) \nu(d\tau) < \infty \quad \text{and} \quad \nu(G - U) < \infty, \tag{1.2}$$

for any Borel neighbourhood U of e.

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2. *For each* $\sigma \in G, f \in C^2(G)$,

$$\begin{aligned} \mathcal{L}f(\sigma) &= b^i X_i f(\sigma) + a^{ij} X_i X_j f(\sigma) \\ &+ \int_G (f(\sigma\tau) - f(\sigma) - x^i(\tau) X_i f(\sigma))\nu(d\tau), \end{aligned} \quad (1.3)$$

where $b = (b^1, \ldots b^n) \in \mathbb{R}^n, a = (a^{ij})$ *is a non-negative-definite, symmetric* $n \times n$ *real-valued matrix and* ν *is a Lévy measure on G.*

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Conversely, any linear operator with a representation as above is the restriction to $C^2(G)$ of the generator corresponding to a unique convolution semigroup of probability measures.

Martingale Representation

Given a càdlàg Lévy process ϕ on G, Applebaum and Kunita (1993) showed that there exists

- A *Brownian motion* $B_a = (B_a(t), t \geq 0)$ on $\mathbb{R}^n$ with covariance
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- An independent Poisson random measure N on $\mathbb{R}^+ \times G$ with intensity measure Leb$\times\nu$ and compensator
$$\tilde{N}(dt, d\sigma) = N(dt, d\sigma) - dt\nu(d\sigma),$$
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so that for all $f \in C^2(G), t \geq 0$

$$\begin{aligned} f(\phi(t)) &= f(e) + \int_0^t \mathcal{L}f(\phi(s-))ds + \int_0^t X_i f(\phi(s-))dB_a^i(s) \\ &+ \int_0^{t+} \int_G (f(\phi(s-)\sigma) - f(\phi(s-))\tilde{N}(ds, d\sigma) \end{aligned} \tag{1.4}$$

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To implement this, choose an $n \times n$ matrix Λ such that $\Lambda\Lambda^T = 2a$ and define $Y_i \in \mathbf{g}$ by $Y_i = \Lambda_i^j X_j$ for $1 \leq i \leq n$.

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$$\begin{aligned}\int_0^t X_i f(\phi(s-)) dB_a^i(s) &= \int_0^t Y_i f(\phi(s-)) dB^i(s) \\ &= \int_0^t \nabla_Y f(\phi(s-)) \cdot dB(s),\end{aligned}$$

where $\nabla_Y := (Y_1, \ldots, Y_n)$ and $\cdot$ is the usual inner product in $\mathbb{R}^n$.

Martingale Transform

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Fix $T > 0$. For each $f \in C_c^\infty(G), \sigma \in G, t \geq 0$, define the *space-time martingale*

$$\begin{aligned} M_f^{(T)}(\sigma, t) &= (P_T f)(\sigma) + \int_0^t \nabla_Y (P_{T-s} f)(\sigma\phi(s-)) \cdot dB(s) \\ &+ \int_0^{t+} \int_G [(P_{T-s} f)(\sigma\phi(s-)\tau) - (P_{T-s} f)(\sigma\phi(s-))] \tilde{N}(ds, d\tau) \end{aligned}$$

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In fact we obtain this when we replace f in (1.4) with $L_\sigma P_{T-\cdot} f$ where $L_\sigma f(\tau) := f(\sigma\tau)$.

Let $A : \mathbb{R}^+ \times G \to M_n(\mathbb{R})$ and $\psi : \mathbb{R}^+ \times G \times G \to \mathbb{R}$ be bounded continuous functions such that $||A|| \vee ||\psi|| \leq 1$ and define the *martingale transform*

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$$\begin{aligned} M_f^{(T;A,\psi)}(\sigma, t) &= \int_0^t A(T - s, \sigma\phi(s-))\nabla_Y(P_{T-s}f)(\sigma\phi(s-)) \cdot dB(s) \\ &+ \int_0^{t+}\int_G \{(P_{T-s}f)(\sigma\phi(s-)\tau) - (P_{T-s}f)(\sigma\phi(s-))\} \\ &\times \{\psi(T - s, \sigma\phi(s-), \tau)\}\tilde{N}(ds, d\tau). \end{aligned} \quad (1.6)$$

Quadratic Variation

$$\begin{aligned}[(M_f^{(T)}(\sigma, \cdot)]_t &= \int_0^t |\nabla_Y (P_{T-s}f)(\sigma\phi(s-))|^2 ds \\ &+ \int_0^{t+}\int_G [(P_{T-s}f)(\sigma\phi(s-)\tau) - (P_{T-s}f)(\sigma\phi(s-))]^2 \\ &\times N(ds, d\tau)\end{aligned}$$

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while

$$[(M_f^{(T;A,\psi)}(\sigma,\cdot)]_t = \int_0^t |A(T-s,\sigma\phi(s-))\nabla_Y(P_{T-s}f)(\sigma\phi(s-))|^2 ds + \int_0^{t+}\int_G \left\{[(P_{T-s}f)(\sigma\phi(s-)\tau) - (P_{T-s}f)(\sigma\phi(s-))]^2\right\} \times \left\{[\psi(T-s,\sigma\phi(s-),\tau)]^2\right\} N(ds, d\tau). \quad (1.8)$$

Apply Burkholders Inequality

From our assumptions on A and ψ we deduce that $(M_f^{(T;A,\psi)}(\sigma, t), 0 \leq t \leq T)$ is differentially subordinate to $(M_f^{(T)}(\sigma, t), t \geq 0)$ and so

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where $M_f^T(\sigma) := M_f^T(\sigma, T)$ and the norm $|| \cdot ||_p$ is in $L^p(\Omega \times G)$ so that

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To obtain (1.9) first apply (1.1) at arbitrary $\sigma \in G$ and then integrate both sides with respect to Haar measure.

But using Fubini's theorem and the right invariance of the Haar measure we see that

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$$\Lambda_g^{T;A,\psi}(f) = \int_G \mathbb{E}(M_f^{(T;A,\psi)}(\sigma)M_g^{(T)}(\sigma))d\sigma \quad (1.11)$$

Using Hölder's inequality, inequality (1.9) and equality (1.10), we obtain

$$|\Lambda_g^{T;A,\psi}(f)| \quad \leq \quad ||M_f^{(T;A,\psi)}||_p ||M_g^{(T)}||_q$$

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$$\begin{aligned} |\Lambda_g^{T;A,\psi}(f)| &\leq ||M_f^{(T;A,\psi)}||_p||M_g^{(T)}||_q \\ &\leq (p^* - 1)||M_f^{(T)}||_p||M_g^{(T)}||_q \end{aligned}$$

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Hence $\Lambda_g^{T;A,\psi}$ extends to a bounded linear functional on $L^p(G)$ and by duality, there exists a bounded linear operator $S_{A,\psi}^T$ on $L^p(G)$ for which

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for all $f \in L^p(G), g \in L^q(G)$ and with

$$||S_{A,\psi}^T||_p \leq (p^* - 1).$$

We can also probe (1.11) using Itô's isometry to find that

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\times & \cdot\nabla_Y(P_{T-s}g)(\sigma\phi(s-))\} ds d\sigma \\
+ & \int_G \int_G \int_0^T \mathbb{E}\{[(P_{T-s}f)(\sigma\phi(s-)\tau) - (P_{T-s}f)(\sigma\phi(s-))] \\
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\times & \cdot\nabla_Y(P_{T-s}g)(\sigma\phi(s-))\}dsd\sigma \\
+ & \int_G \int_G \int_0^T \mathbb{E}\{[(P_{T-s}f)(\sigma\phi(s-)\tau) - (P_{T-s}f)(\sigma\phi(s-))] \\
\times & [(P_{T-s}g)(\sigma\phi(s-)\tau) - (P_{T-s}g)(\sigma\phi(s-))] \\
\times & \psi(T-s, \sigma\phi(s-), \tau)\}ds\nu(d\tau)d\sigma \\
= & \int_0^T \int_G A(T-s, \sigma)\nabla_Y(P_{T-s}f)(\sigma) \cdot \nabla_Y(P_{T-s}g)(\sigma)d\sigma ds \\
+ & \int_0^T \int_G \int_G [(P_{T-s}f)(\sigma\tau) - (P_{T-s}f)(\sigma)][(P_{T-s}g)(\sigma\tau) - (P_{T-s}g)(\sigma)] \\
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\end{aligned}
$$

and so

$$\begin{aligned}\Lambda_g^{T;A,\psi}(f) &= \int_0^T \int_G A(s,\sigma)\nabla_Y(P_s f)(\sigma) \cdot \nabla_Y(P_s g)(\sigma) d\sigma ds \\ &+ \int_0^T \int_G \int_G [(P_s f)(\sigma\tau) - (P_s f)(\sigma)][(P_s g)(\sigma\tau) - (P_s g)(\sigma)] \\ &\times \psi(s,\sigma,\tau)\nu(d\tau) d\sigma ds. \end{aligned} \tag{1.13}$$

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$$

Choosing

$$A(s,\sigma) = \frac{\nabla_Y(P_s f)(\sigma)\otimes\nabla_Y(P_s g)(\sigma)}{|\nabla_Y(P_s f)(\sigma)|\,|\nabla_Y(P_s g)(\sigma)|},$$

and

$$\psi(s,\sigma,\tau) = \text{sign}((P_s f)(\sigma\tau) - P_s f(\sigma))((P_s g)(\sigma\tau) - P_s g(\sigma)))$$

we can show that the integrals on the RHS of (1.13) converge absolutely. Hence we can apply the previous argument in a time-independent manner to conclude the following.

Construction of Bounded Linear Operators on $L^p(G)$

Theorem

Let $A : \mathbb{R}^+ \times G \to M_n(\mathbb{R})$ *and* $\psi : \mathbb{R}^+ \times G \times G \to M_n(\mathbb{R})$ *be bounded continuous functions such that* $||A|| \vee ||\psi|| \leq 1$. *There exists a bounded linear operator* $S_{A,\psi}$ *on* $L^p(G)$, $1 < p < \infty$, *for which*

$$\begin{aligned}
& \int_G S_{A,\psi} f(\sigma) g(\sigma) d\sigma \\
= & \int_0^\infty \int_G A(s,\sigma) \nabla_Y (P_s f)(\sigma) \cdot \nabla_Y (P_s g)(\sigma) d\sigma ds \\
+ & \int_0^\infty \int_G \int_G [(P_s f)(\sigma\tau) - (P_s f)(\sigma)] \\
\times & [(P_s g)(\sigma\tau) - (P_s g)(\sigma)] \psi(s,\sigma,\tau) \nu(d\tau) d\sigma ds, \qquad (1.14)
\end{aligned}$$

for all $f, g \in C_c^\infty(G)$.

Theorem

(continued) Furthermore, for all $f \in L^p(G)$ *and* $g \in L^q(G)$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left| \int_G S_{A,\psi} f(\sigma) g(\sigma) d\sigma \right| \leq (p^* - 1)\|f\|_p \, \|g\|_q \tag{1.15}$$

and

$$\|S_{A,\psi} f\|_p \leq (p^* - 1)\|f\|_p. \tag{1.16}$$

To exhibit $S_{A,\psi}$ as a Fourier multiplier, we need to apply Plancherel's theorem within (1.14).

Compact Lie Groups: Representations, Peter-Weyl Theorem

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We define the co-ordinate functions $\pi_{ij}(\sigma) = \pi(\sigma)_{ij}$ for $\sigma \in G, 1 \leq i, j \leq d_\pi$. We have $\pi_{ij} \in C^\infty(G)$.

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Theorem (Peter-Weyl)

$\{\sqrt{d_\pi}\pi_{ij}; 1 \leq i, j \leq d_\pi, \pi \in \widehat{G}\}$ *is a complete orthonormal basis for* $L^2(G, \mathbb{C})$.

Fourier Multipliers

For each $f \in L^2(G, \mathbb{C})$, we define its *non-commutative Fourier transform* to be the matrix $\widehat{f}(\pi)$ defined by

$$\widehat{f}(\pi)_{ij} = \int_G f(\sigma)\pi_{ij}(\sigma^{-1})d\sigma,$$

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$$\int_G f(\sigma)\overline{g(\sigma)}d\sigma = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(\widehat{f}(\pi)\widehat{g}(\pi)^*), \quad (1.17)$$

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for $f, g \in L^2(G, \mathbb{C})$. In particular if T is a bounded linear operator on $L^2(G, \mathbb{C})$ we have

$$\int_G Tf(\sigma)\overline{g(\sigma)}d\sigma = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(\widehat{Tf}(\pi)\widehat{g}(\pi)^*). \qquad (1.18)$$

We say that the operator T is a *Fourier multiplier* if for each $\pi \in \widehat{G}$ there exists a $d_\pi \times d_\pi$ complex matrix $m_T(\pi)$ so that

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We call the matrices $(m_T(\pi), \pi \in \widehat{G})$ the *symbol* of the operator T.

Casimir Spectrum

Given $\pi \in \widehat{G}$ we obtain the *derived representation* $d\pi$ of the Lie algebra **g** from the identity

$$\pi(\exp(X)) = e^{d\pi(X)},$$

for each $X \in \mathbf{g}$.

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Then $\Omega_\pi = -\kappa_\pi I_\pi$ where I_π is the identity matrix acting on V_π and $\kappa_\pi \geq 0$.

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The Laplace-Beltrami operator $\Delta = \sum_{i=1}^n X_i^2$ is an essentially self-adjoint operator in $L^2(G, \mathbb{C})$ with domain $C^\infty(G, \mathbb{C})$ having discrete spectrum with

$$\Delta \pi_{ij} = -\kappa_\pi \pi_{ij}, \qquad (1.20)$$

for all $\pi \in \widehat{G}, 1 \leq i, j \leq d_\pi$.

Then the heat semigroup $P_t = e^{t\Delta}$ satisfies

$$P_t \pi_{ij} = e^{-t\kappa_\pi} \pi_{ij}, \quad (1.21)$$

for all $t \geq 0, \pi \in \widehat{G}, 1 \leq i, j \leq d_\pi$.

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Hence for $f \in L^2(G)$,

$$\widehat{P_t f} = e^{-t\kappa_\pi} \widehat{f} \qquad (1.22)$$

Operators of Laplace-Transform Type

Put $\psi = 0$ and $a = 2I$ so ϕ is a Brownian motion with generator Δ. Let $A = A(t)$ be a function only of time.

Then (1.14), integration by parts and (1.18) yields for all $f, g \in C^{\infty}(G)$,

$$\int_G S_A f(\sigma) g(\sigma) d\sigma \quad = \quad 2 \int_0^{\infty} \int_G A(s) \nabla_X (P_s f)(\sigma) \cdot \nabla_X (P_s g)(\sigma) d\sigma ds,$$

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Hence for all $\pi \in \widehat{G}$ we have

$$\widehat{S_A f}(\pi) = \left(\int_0^\infty 2\kappa_\pi e^{-2s\kappa_\pi} A(s) ds\right) \widehat{f}(\pi)$$

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and S_A is an operator of *Laplace transform-type*. It is clearly a Fourier multiplier in the sense of (1.19).

A special case of particular interest is obtained by taking $A(s) = \dfrac{(2s)^{-i\gamma}}{\Gamma(1 - i\gamma)} I$, where $\gamma \in \mathbb{R}$.

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Corollary

For $1 < p < \infty$, $f \in L^p(G)$, we have

$$\|(-\Delta)^{i\gamma} f\|_p \leq \frac{p^* - 1}{|\Gamma(1 - i\gamma)|} \|f\|_p.$$

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For $1 < p < \infty$, $f \in L^p(G)$, *we have*

$$||(-\Delta)^{i\gamma} f||_p \leq \frac{p^* - 1}{|\Gamma(1 - i\gamma)|} ||f||_p. \tag{1.24}$$

We have many examples when $A = 0$ and the contribution is from the non-local part of the process. One class of such examples is obtained by *subordination of Brownian motion.*

Multipliers via Subordination

In this section we will consider an operator of the form $S_{\psi} := S_{0,\psi}$ which is built from the non-local part of the Lévy process.

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Then we have

$$\mathbb{E}(e^{-uT(t)}) = e^{-th(u)}$$

for all $u > 0, t \geq 0$ where $h : (0, \infty) \to \mathbb{R}^+$ is a Bernstein function for which $\lim_{u \to 0} h(u) = 0$.

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Hence h must take the form

$$h(u) = cu + \int_{(0,\infty)} (1 - e^{-uy})\lambda(dy) \tag{1.25}$$

where $c \geq 0$ and λ is a Borel measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge y)\lambda(dy) < \infty$.

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It is well-known that if ϕ is a Lévy process in G and $(T(t), t \geq 0)$ is an independent subordinator with Bernstein function h then the process $\phi_T = (\phi_T(t), t \geq 0)$ is again a Lévy process in G where

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$$P_t^h f(\sigma) = \int_{(0,\infty)} P_s f(\sigma) \rho_t(ds) \tag{1.26}$$

for all $t \geq 0, f \in C(G), \sigma \in G$.

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We denote the law of $T(t)$ by ρ_t, for each $t \geq 0$.

Let $(P_t, t \geq 0)$ be the semigroup of convolution operators on $C(G)$ that is associated to the process ϕ and $(P_t^h, t \geq 0)$ be the corresponding semigroup of convolution operators associated to the process ϕ_T. We have

$$P_t^h f(\sigma) = \int_{(0,\infty)} P_s f(\sigma) \rho_t(ds) \qquad (1.26)$$

for all $t \geq 0, f \in C(G), \sigma \in G$. In the sequel we will always take ϕ to be Brownian motion on G.

Theorem

If $(P_t, t \geq 0)$ *is the heat semigroup on G then for all* $f \in C(G), \pi \in \widehat{G}, t \geq 0$,

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We now take $A = 0$ in (1.14) and take the Lévy process to be of the form ϕ_T as just described. For simplicity we also assume that ψ only depends on the jumps of the process and so we write $\psi(\tau) := \psi(\cdot, \cdot, \tau)$ for each $\tau \in G$.

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$$\int_G S_\psi f(\sigma) g(\sigma) d\sigma = \int_0^\infty \int_G \int_G [(P_s^h f)(\sigma\tau) - (P_s^h f)(\sigma)] \times [(P_s^h g)(\sigma\tau) - (P_s^h g)(\sigma)] \psi(\tau) \nu(d\tau) d\sigma ds \quad (1.27)$$

Now using (1.18) and Proposition 7 we obtain

$$
\begin{aligned}
& \int_G S_\psi f(\sigma) g(\sigma) d\sigma \\
= & \int_0^\infty \int_G \sum_{\pi \in \widehat{G}} d_\pi e^{-2sh(\kappa_\pi)} \mathrm{tr}[(\pi(\tau) - I_\pi)\widehat{f}(\pi)\widehat{g}(\pi)^*(\pi(\tau)^* - I_\pi)] \\
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\end{aligned}$$

and so S_ψ is a Fourier multiplier with

$$m_{S_\psi}(\pi) = \frac{1}{2h(\kappa_\pi)} \int_G (2I_\pi - \pi(\tau) - \pi(\tau)^*)\psi(\tau)\nu(d\tau), \qquad (1.28)$$

for $\pi \in \widehat{G}$.

From Theorem 3 we obtain

Corollary

Let S_ψ *be the operator with Fourier multiplier given by* (1.28). *If* $\|\psi\| \leq 1$, *then*

$$\|S_\psi f\|_p \leq (p^* - 1)\|f\|_p, \tag{1.29}$$

for all $1 < p < \infty$, $f \in L^p(G)$.

From Theorem 3 we obtain

Corollary

Let S_ψ be the operator with Fourier multiplier given by (1.28)*. If* $||\psi|| \leq 1$*, then*

$$||S_\psi f||_p \leq (p^* - 1)||f||_p, \tag{1.29}$$

for all $1 < p < \infty$*,* $f \in L^p(G)$*.*

We can also obtain examples of multipliers generated by Lévy processes on $\mathbb{R}^n$ as in

R.Bañuelos, K.Bogdan, *J. Funct. Anal.* **250**, 197(2007) and

R.Bañuelos, A.Bielaszewski, K.Bogdan, Marcinkiewcz Centenary Volume, *Banach Center Publications*, **95**, 9(2012).

Vielen Dank für Ihre Aufmerksamkeit.

Thank you for listening.