Lectures on Lévy Processes and Stochastic Calculus, Braunschweig, Lecture 2: Lévy Processes

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Definition: Lévy Process

Let $X = (X(t), t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$.

We say that it has *independent increments* if for each $n \in \mathbb{N}$ and each $0 \leq t_1 < t_2 < \cdots < t_{n+1} < \infty$, the random variables $(X(t_{j+1}) - X(t_j), 1 \leq j \leq n)$ are independent

and it has *stationary increments* if each $X(t_{j+1}) - X(t_j) \stackrel{d}{=} X(t_{j+1} - t_j) - X(0)$.

We say that X is a *Lévy process* if

(L1) Each $X(0) = 0$ (a.s),

(L2) X has independent and stationary increments,

(L3) X is *stochastically continuous* i.e. for all $a > 0$ and for all $s \geq 0$,

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Note that in the presence of (L1) and (L2), (L3) is equivalent to the condition

$$\lim_{t \downarrow 0} P(|X(t)| > a) = 0.$$

The *sample paths* of a process are the maps $t \to X(t)(\omega)$ from $\mathbb{R}^+$ to $\mathbb{R}^d$, for each $\omega \in \Omega$.

We are now going to explore the relationship between Lévy processes and infinite divisibility.

Theorem

If X is a Lévy process, then $X(t)$ is infinitely divisible for each $t \geq 0$.

Proof. For each $n \in \mathbb{N}$, we can write

$$X(t) = Y_1^{(n)}(t) + \cdots + Y_n^{(n)}(t)$$

where each $Y_k^{(n)}(t) = X(\frac{kt}{n}) - X(\frac{(k-1)t}{n})$.
The $Y_k^{(n)}(t)$'s are i.i.d. by (L2). □

From Lecture 1 we can write $\phi_{X(t)}(u) = e^{\eta(t,u)}$ for each $t \geq 0, u \in \mathbb{R}^d$, where each $\eta(t, \cdot)$ is a Lévy symbol.

Theorem

If X is a Lévy process, then

$$\phi_{X(t)}(u) = e^{t\eta(u)},$$

for each $u \in \mathbb{R}^d, t \geq 0$, where η is the Lévy symbol of $X(1)$.

Proof. Suppose that X is a Lévy process and for each $u \in \mathbb{R}^d, t \geq 0$, define $\phi_u(t) = \phi_{X(t)}(u)$
then by (L2) we have for all $s \geq 0$,

$$\begin{aligned} \phi_u(t+s) &= \mathbb{E}(e^{i(u,X(t+s))}) \\ &= \mathbb{E}(e^{i(u,X(t+s)-X(s))}e^{i(u,X(s))}) \\ &= \mathbb{E}(e^{i(u,X(t+s)-X(s))})\mathbb{E}(e^{i(u,X(s))}) \\ &= \phi_u(t)\phi_u(s) \ldots \text{(i)} \end{aligned}$$

X

Now $\phi_u(0) = 1 \ldots$ (ii) by (L1), and the map $t \to \phi_u(t)$ is continuous. However the unique continuous solution to (i) and (ii) is given by $\phi_u(t) = e^{t\alpha(u)}$, where $\alpha : \mathbb{R}^d \to \mathbb{C}$. Now by Theorem 1, $X(1)$ is infinitely divisible, hence α is a Lévy symbol and the result follows.
□

We now have the Lévy-Khinchine formula for a Lévy process $X = (X(t), t \geq 0)$:-

$$\begin{aligned} \mathbb{E}(e^{i(u,X(t))}) &= \exp\{\left(t\left[i(b,u) - \frac{1}{2}(u,Au)\right.\right. \\ &\left.\left. + \int_{\mathbb{R}^d-\{0\}} (e^{i(u,y)} - 1 - i(u,y)\mathbf{1}_{\hat{B}}(y))\nu(dy)\right]\right)\}, \quad (2.1) \end{aligned}$$

for each $t \geq 0, u \in \mathbb{R}^d$, where (b, A, ν) are the characteristics of $X(1)$.
We will define the Lévy symbol and the characteristics of a Lévy process X to be those of the random variable $X(1)$. We will sometimes write the former as η_X when we want to emphasise that it belongs to the process X.

Let p_t be the law of $X(t)$, for each $t \geq 0$. By (L2), we have for all $s, t \geq 0$ that:

$$p_{t+s} = p_t * p_s.$$

By (L3), we have $p_t \xrightarrow{w} \delta_0$ as $t \to 0$, i.e. $\lim_{t \to 0} f(x)p_t(dx) = f(0)$.

$(p_t, t \geq 0)$ is a *weakly continuous convolution semigroup of probability measures* on $\mathbb{R}^d$.

Conversely, given any such semigroup, we can always construct a Lévy process on path space via Kolmogorov's construction.

Informally, we have the following asymptotic relationship between the law of a Lévy process and its Lévy measure:

$$\nu = \lim_{t \downarrow 0} \frac{p_t}{t}.$$

More precisely

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} f(x)p_t(dx) = \int_{\mathbb{R}^d} f(x)\nu(dx), \qquad (2.2)$$

for bounded, continuous functions f which vanish in some neighborhood of the origin.

Examples of Lévy Processes

Example 1, Brownian Motion and Gaussian Processes

A *(standard) Brownian motion* in $\mathbb{R}^d$ is a Lévy process $B = (B(t), t \geq 0)$ for which

(B1) $B(t) \sim N(0, tI)$ for each $t \geq 0$,
(B2) B has continuous sample paths.

It follows immediately from (B1) that if B is a standard Brownian motion, then its characteristic function is given by

$$\phi_{B(t)}(u) = \exp\{-\frac{1}{2}t|u|^2\},$$

for each $u \in \mathbb{R}^d, t \geq 0$.

We introduce the marginal processes $B_i = (B_i(t), t \geq 0)$ where each $B_i(t)$ is the ith component of $B(t)$, then it is not difficult to verify that the B_i's are mutually independent Brownian motions in $\mathbb{R}$. We will call these *one-dimensional Brownian motions* in the sequel.
Brownian motion has been the most intensively studied Lévy process. In the early years of the twentieth century, it was introduced as a model for the physical phenomenon of Brownian motion by Einstein and Smoluchowski and as a description of the dynamical evolution of stock prices by Bachelier.

The theory was placed on a rigorous mathematical basis by Norbert Wiener in the 1920's.

We could try to use the Kolmogorov existence theorem to construct one-dimensional Brownian motion from the following prescription on cylinder sets of the form $I^H_{t_1,\ldots,t_n} = \{\omega \in \Omega; \omega(t_1) \in [a_1, b_1], \ldots, \omega(t_n) \in [a_n, b_n]\}$ where $H = [a_1, b_1] \times \cdots [a_n, b_n]$ and we have taken Ω to be the set of all mappings from $\mathbb{R}^+$ to $\mathbb{R}$:

$$\begin{aligned} & P(I^H_{t_1,\ldots,t_n}) \\ = & \int_H \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{t_1(t_2-t_1)\ldots(t_n-t_{n-1})}} \exp\left(-\frac{1}{2}\left(\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \cdots\right.\right. \\ + & \left.\left.\frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\right)\right) dx_1 \cdots dx_n. \end{aligned}$$

However there there is then no guarantee that the paths are continuous.

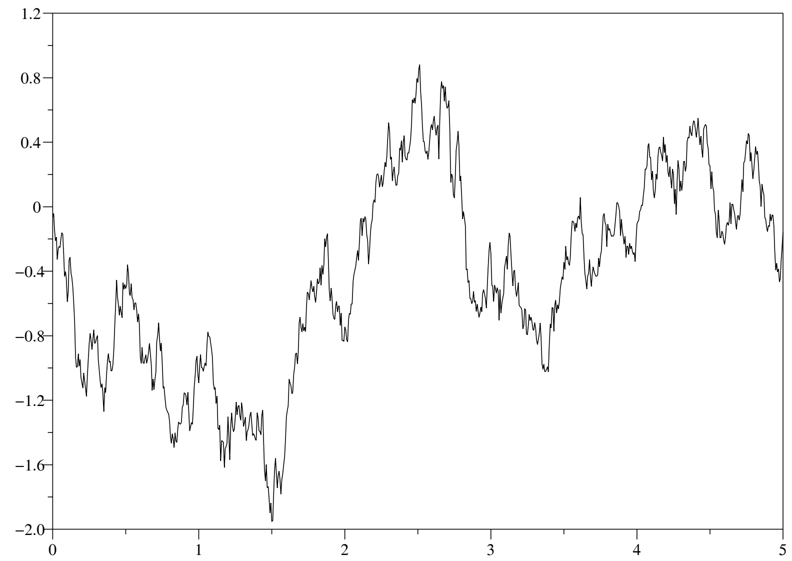
The literature contains a number of ingenious methods for constructing Brownian motion. One of the most delightful of these (originally due to Paley and Wiener) obtains this, in the case $d = 1$, as a random Fourier series for $0 \leq t \leq 1$:

$$B(t) = \frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\frac{\sin(\pi t(n+\frac{1}{2}))}{n+\frac{1}{2}}\xi(n),$$

for each $t \geq 0$, where $(\xi(n), n \in \mathbb{N})$ is a sequence of i.i.d. N$(0,1)$ random variables.

We list a number of useful properties of Brownian motion in the case $d = 1$.

- Brownian motion is locally Hölder continuous with exponent α for every $0 < \alpha < \frac{1}{2}$ i.e. for every $T > 0, \omega \in \Omega$ there exists $K = K(T, \omega)$ such that
$$|B(t)(\omega) - B(s)(\omega)| \leq K|t-s|^{\alpha},$$
for all $0 \leq s < t \leq T$.
- The sample paths $t \rightarrow B(t)(\omega)$ are almost surely nowhere differentiable.
- For any sequence, $(t_n, n \in \mathbb{N})$ in $\mathbb{R}^+$ with $t_n \uparrow \infty$,
$$\liminf_{n\rightarrow\infty} B(t_n) = -\infty \text{ a.s.} \quad \limsup_{n\rightarrow\infty} B(t_n) = \infty \text{ a.s.}$$
- The law of the iterated logarithm:-
$$P\left(\limsup_{t\downarrow 0}\frac{B(t)}{(2t\log(\log(\frac{1}{t})))^{\frac{1}{2}}} = 1\right) = 1.$$



Simulation of standard Brownian motion

Let A be a non-negative symmetric $d \times d$ matrix and let σ be a square root of A so that σ is a $d \times m$ matrix for which $\sigma\sigma^T = A$. Now let $b \in \mathbb{R}^d$ and let B be a Brownian motion in R^m. We construct a process $C = (C(t), t \geq 0)$ in $\mathbb{R}^d$ by

$$C(t) = bt + \sigma B(t), \tag{2.3}$$

then C is a Lévy process with each $C(t) \sim N(tb, tA)$. It is not difficult to verify that C is also a Gaussian process, i.e. all its finite dimensional distributions are Gaussian. It is sometimes called *Brownian motion with drift*. The Lévy symbol of C is

$$\eta_C(u) = i(b, u) - \frac{1}{2}(u, Au).$$

In fact a Lévy process has continuous sample paths if and only if it is of the form (2.3).

Example 2 - The Poisson Process

The Poisson process of intensity $\lambda > 0$ is a Lévy process N taking values in $\mathbb{N} \cup \{0\}$ wherein each $N(t) \sim \pi(\lambda t)$ so we have

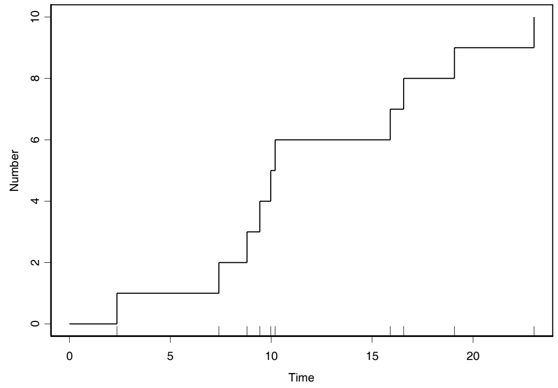
$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

for each $n = 0, 1, 2, \ldots$.
The Poisson process is widely used in applications and there is a wealth of literature concerning it and its generalisations.

We define non-negative random variables $(T_n, \mathbb{N} \cup \{0\})$ (usually called waiting times) by $T_0 = 0$ and for $n \in \mathbb{N}$,

$$T_n = \inf\{t \geq 0; N(t) = n\},$$

then it is well known that the T_n's are gamma distributed. Moreover, the inter-arrival times $T_n - T_{n-1}$ for $n \in \mathbb{N}$ are i.i.d. and each has exponential distribution with mean $\frac{1}{\lambda}$. The sample paths of N are clearly piecewise constant with "jump" discontinuities of size 1 at each of the random times $(T_n, n \in \mathbb{N})$.



Simulation of a Poisson process ($\lambda = 0.5$)

For later work it is useful to introduce the *compensated Poisson process* $\tilde{N} = (\tilde{N}(t), t \geq 0)$ where each $\tilde{N}(t) = N(t) - \lambda t$. Note that $\mathbb{E}(\tilde{N}(t)) = 0$ and $\mathbb{E}(\tilde{N}(t)^2) = \lambda t$ for each $t \geq 0$.

Example 3 - The Compound Poisson Process

Let $(Z(n), n \in \mathbb{N})$ be a sequence of i.i.d. random variables taking values in $\mathbb{R}^d$ with common law μ_Z and let N be a Poisson process of intensity λ which is independent of all the $Z(n)$'s. The *compound Poisson process* Y is defined as follows:-

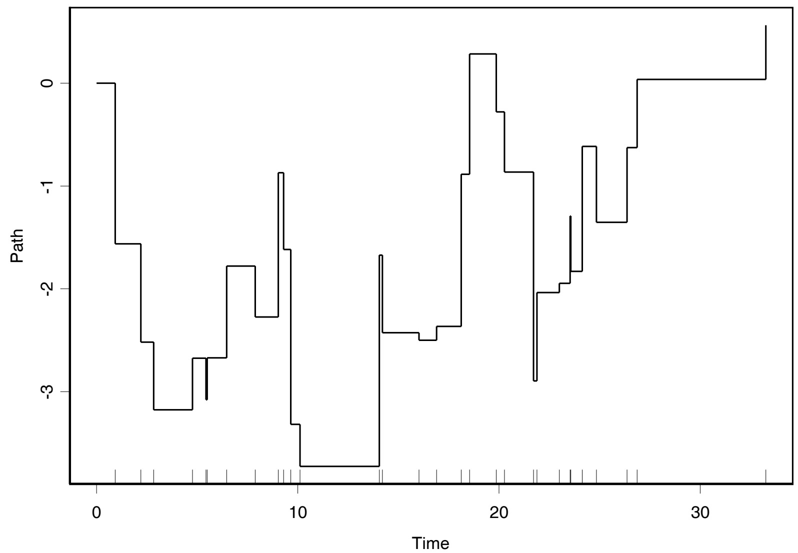
$$Y(t) := \begin{cases} 0 & \text{if } N(t) = 0 \\ Z(1) + \cdots + Z(N(t)) & \text{if } N(t) > 0, \end{cases}$$

for each $t \geq 0$,
so each $Y(t) \sim \pi(\lambda t, \mu_Z)$.

From the work of Lecture 1, Y has Lévy symbol

$$\eta_Y(u) = \left[\int (e^{i(u,y)} - 1) \lambda \mu_Z(dy) \right].$$

Again the sample paths of Y are piecewise constant with "jump discontinuities" at the random times $(T(n), n \in \mathbb{N})$, however this time the size of the jumps is itself random, and the jump at $T(n)$ can be any value in the range of the random variable $Z(n)$.



Simulation of a compound Poisson process with $N(0,1)$ summands($\lambda = 1$).

Example 4 - Interlacing Processes

Let C be a Gaussian Lévy process as in Example 1 and Y be a compound Poisson process as in Example 3, which is independent of C.
Define a new process X by

$$X(t) = C(t) + Y(t),$$

for all $t \geq 0$, then it is not difficult to verify that X is a Lévy process with Lévy symbol

$$\eta_X(u) = i(b,u) - \frac{1}{2}(u, Au) + \left[\int (e^{i(u,y)} - 1)\lambda\mu_Z(dy)\right].$$

Using the notation of Examples 2 and 3, we see that the paths of X have jumps of random size occurring at random times.

$$\begin{aligned} X(t) &= C(t) \quad \text{for } 0 \leq t < T_1, \\ &= C(T_1) + Z_1 \quad \text{when } t = T_1, \\ &= X(T_1) + C(t) - C(T_1) \quad \text{for } T_1 < t < T_2, \\ &= X(T_2-) + Z_2 \quad \text{when } t = T_2, \end{aligned}$$

and so on recursively. We call this procedure an *interlacing* as a continuous path process is “interlaced” with random jumps. It seems reasonable that the most general Lévy process might arise as the limit of a sequence of such interlacings, and this can be established rigorously.

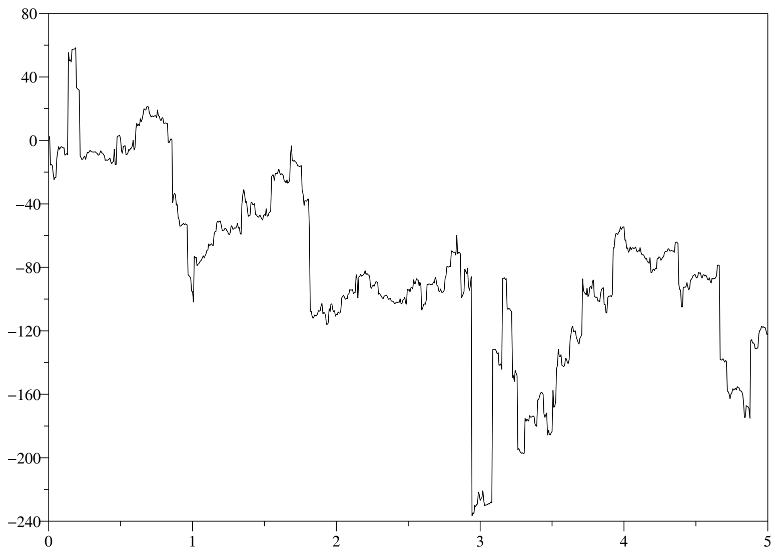
Example 5 - Stable Lévy Processes

A *stable Lévy process* is a Lévy process X in which the Lévy symbol is that of a given stable law. So, in particular, each $X(t)$ is a stable random variable. For example, we have the rotationally invariant case whose Lévy symbol is given by

$$\eta(u) = -\sigma^\alpha |u|^\alpha,$$

where α is the index of stability ($0 < \alpha \leq 2$). One of the reasons why these are important in applications is that they display self-similarity.

In general, a stochastic process $Y = (Y(t), t \geq 0)$ is *self-similar with Hurst index* $H > 0$ if the two processes $(Y(at), t \geq 0)$ and $(a^H Y(t), t \geq 0)$ have the same finite-dimensional distributions for all $a \geq 0$. By examining characteristic functions, it is easily verified that a rotationally invariant stable Lévy process is self-similar with Hurst index $H = \frac{1}{\alpha}$, so that e.g. Brownian motion is self-similar with $H = \frac{1}{2}$. A Lévy process X is self-similar if and only if each $X(t)$ is strictly stable.



Simulation of the Cauchy process.

Densities of Lévy Processes

Question: When does a Lévy process have a density f_t for all $t > 0$ so that for all Borel sets B:

$$P(X_t \in B) = p_t(B) = \int_B f_t(x)dx?$$

In general, a random variable has a continuous density if its characteristic function is integrable and in this case, the density is the Fourier transform of the characteristic function.
So for Lévy processes, if for all $t > 0$,

$$\int_{\mathbb{R}^d} |e^{t\eta(u)}|du = \int_{\mathbb{R}^d} e^{t\Re(\eta(u))}du < \infty$$

we then have

$$f_t(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{t\eta(u)-i(u,x)}du.$$

Every Lévy process with a non-degenerate Gaussian component has a density.

In this case

$$\Re(\eta(u)) = -\frac{1}{2}(u, Au) + \int_{\mathbb{R}^d-\{0\}}(\cos(u,y) - 1)\nu(dy),$$

and so

$$\int_{\mathbb{R}^d} e^{t\Re(\eta(u))}du \leq \int_{\mathbb{R}^d} e^{-\frac{t}{2}(u,Au)} < \infty,$$

using $(u, Au) \geq \lambda|u|^2$ where $\lambda > 0$ is smallest eigenvalue of A.

For examples where densities exist for $A = 0$ with $d = 1$: if X is α-stable, it has a density since for all $1 \leq \alpha \leq 2$:

$$\int_{|u|\geq 1} e^{-t|u|^\alpha}du \leq \int_{|u|\geq 1} e^{-t|u|}du < \infty,$$

and for $0 \leq \alpha < 1$:

$$\int_{\mathbb{R}} e^{-t|u|^\alpha}du = \frac{2}{\alpha}\int_0^\infty e^{-ty}y^{\frac{1}{\alpha}-1}dy < \infty.$$

In general, a sufficient condition for a density is

- $\nu(\mathbb{R}^d) = \infty$
- $\tilde{\nu}^{*m}$ is absolutely continuous with respect to Lebesgue measure for some $m \in \mathbb{N}$ where

$$\tilde{\nu}(A) = \int_A (|x|^2 \wedge 1)\nu(dx).$$

A Lévy process has a *Lévy density* g_ν if its Lévy measure ν is absolutely continuous with respect to Lebesgue measure, then g_ν is defined to be the Radon-Nikodym derivative $\frac{d\nu}{dx}$.

A process may have a Lévy density but not have a density.

Example. Let X be a compound Poisson process with each $X(t) = Y_1 + Y_2 + \cdots + Y_{N(t)}$ wherein each Y_j has a density f_Y, then $g_\nu = \lambda f_Y$ is the Lévy density.

But

$$P(Y(t) = 0) \geq P(N(t) = 0) = e^{-\lambda t} > 0,$$

so p_t has an atom at $\{0\}$.

We have $p_t(A) = e^{-\lambda t}\delta_0(A) + \int_A f_t^{ac}(x)dx$, where for $x \neq 0$

$$f_t^{ac}(x) = e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}f_Y^{*n}(x).$$

$f_t^{ac}(x)$ is the conditional density of $X(t)$ given that it jumps at least once between 0 and t.

In this case, (2.2) takes the precise form (for $x \neq 0$)

$$g_\nu(x) = \lim_{t \downarrow 0}\frac{f_t^{ac}(x)}{t}.$$

Subordinators

A *subordinator* is a one-dimensional Lévy process which is increasing a.s. Such processes can be thought of as a random model of time evolution, since if $T = (T(t), t \geq 0)$ is a subordinator we have

$$T(t) \geq 0 \text{ for each } t > 0 \text{ a.s.} \quad \text{and } T(t_1) \leq T(t_2) \text{ whenever } t_1 \leq t_2 \text{ a.s.}$$

Now since for $X(t) \sim N(0, At)$ we have $P(X(t) \geq 0) = P(X(t) \leq 0) = \frac{1}{2}$, it is clear that such a process cannot be a subordinator.

Theorem

If T is a subordinator then its Lévy symbol takes the form

$$\eta(u) = ibu + \int_{(0,\infty)} (e^{iuy} - 1)\lambda(dy), \qquad (2.4)$$

where $b \geq 0$, and the Lévy measure λ satisfies the additional requirements

$$\lambda(-\infty, 0) = 0 \ \text{ and } \int_{(0,\infty)} (y \wedge 1)\lambda(dy) < \infty.$$

Conversely, any mapping from $\mathbb{R}^d \to \mathbb{C}$ of the form (2.4) is the Lévy symbol of a subordinator.

We call the pair (b, λ), the *characteristics* of the subordinator T.

For each $t \geq 0$, the map $u \to \mathbb{E}(e^{iuT(t)})$ can be analytically continued to the region $\{iu, u > 0\}$ and we then obtain the following expression for the Laplace transform of the distribution

$$\mathbb{E}(e^{-uT(t)}) = e^{-t\psi(u)},$$

$$\text{where } \psi(u) = -\eta(iu) = bu + \int_{(0,\infty)} (1 - e^{-uy})\lambda(dy) \qquad (2.5)$$

for each $t, u \geq 0$.

This is much more useful for both theoretical and practical application than the characteristic function.

The function ψ is usually called the *Laplace exponent* of the subordinator.

Examples of Subordinators

(1) **The Poisson Case**

Poisson processes are clearly subordinators. More generally a compound Poisson process will be a subordinator if and only if the $Z(n)$'s are all $\mathbb{R}^+$ valued.

(2) α**-Stable Subordinators**

Using straightforward calculus, we find that for $0 < \alpha < 1$, $u \geq 0$,

$$u^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-ux}) \frac{dx}{x^{1+\alpha}}.$$

Hence for each $0 < \alpha < 1$ there exists an α-stable subordinator T with Laplace exponent

$$\psi(u) = u^\alpha.$$

and the characteristics of T are $(0, \lambda)$ where $\lambda(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dx}{x^{1+\alpha}}$.

Note that when we analytically continue this to obtain the Lévy symbol we obtain the form given in Lecture 1 for stable laws with $\mu = 0, \beta = 1$ and $\sigma^\alpha = \cos\left(\frac{\alpha\pi}{2}\right)$.

(3) **The Lévy Subordinator**

The $\frac{1}{2}$-stable subordinator has a density given by the Lévy distribution (with $\mu = 0$ and $\sigma = \frac{t^2}{2}$)

$$f_{T(t)}(s) = \left(\frac{t}{2\sqrt{\pi}}\right) s^{-\frac{3}{2}} e^{-\frac{t^2}{4s}},$$

for $s \geq 0$. The Lévy subordinator has a nice probabilistic interpretation as a first hitting time for one-dimensional standard Brownian motion $(B(t), t \geq 0)$,

$$T(t) = \inf\left\{s > 0; B(s) = \frac{t}{\sqrt{2}}\right\}. \tag{2.6}$$

To show directly that for each $t \geq 0$,

$$\mathbb{E}(e^{-uT(t)}) = \int_0^\infty e^{-us} f_{T(t)}(s)ds = e^{-tu^{\frac{1}{2}}},$$

write $g_t(u) = \mathbb{E}(e^{-uT(t)})$. Differentiate with respect to u and make the substitution $x = \frac{t^2}{4us}$ to obtain the differential equation $g_t'(u) = -\frac{t}{2\sqrt{u}} g_t(u)$. Via the substitution $y = \frac{t}{2\sqrt{s}}$ we see that $g_t(0) = 1$ and the result follows.

(4) **Inverse Gaussian Subordinators**

We generalise the Lévy subordinator by replacing Brownian motion by the Gaussian process $C = (C(t), t \geq 0)$ where each $C(t) = B(t) + \mu t$ and $\mu \in \mathbb{R}$. The *inverse Gaussian subordinator* is defined by

$$T(t) = \inf\{s > 0; C(s) = \delta t\}$$

where $\delta > 0$ and is so-called since $t \to T(t)$ is the generalised inverse of a Gaussian process.
Using martingale methods, we can show that for each $t, u > 0$,

$$\mathbb{E}(e^{-uT(t)}) = e^{-t\delta(\sqrt{2u+\mu^2}-\mu)}, \tag{2.7}$$

In fact each $T(t)$ has a density:-

$$f_{T(t)}(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t \mu} s^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}(t^2\delta^2 s^{-1} + \mu^2 s)\right\}, \tag{2.8}$$

for each $s, t \geq 0$.
In general any random variable with density $f_{T(1)}$ is called an *inverse Gaussian* and denoted as IG(δ, μ).

(5) **Gamma Subordinators**

Let $(T(t), t \geq 0)$ be a *gamma process* with parameters $a, b > 0$ so that each $T(t)$ has density

$$f_{T(t)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx},$$

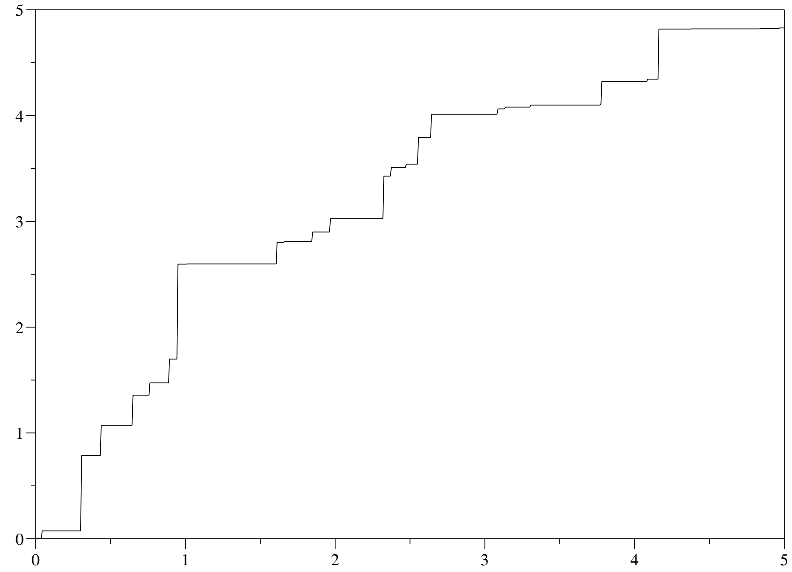
for $x \geq 0$; then it is easy to verify that for each $u \geq 0$,

$$\int_0^\infty e^{-ux} f_{T(t)}(x)dx = \left(1 + \frac{u}{b}\right)^{-at} = \exp\left(-ta\log\left(1 + \frac{u}{b}\right)\right).$$

From here it is a straightforward exercise in calculus to show that

$$\int_0^\infty e^{-ux} f_{T(t)}(x)dx = \exp\left[-t\int_0^\infty (1 - e^{-ux})ax^{-1}e^{-bx}dx\right].$$

From this we see that $(T(t), t \geq 0)$ is a subordinator with $b = 0$ and $\lambda(dx) = ax^{-1}e^{-bx}dx$. Moreover $\psi(u) = a\log\left(1 + \frac{u}{b}\right)$ is the associated Bernstein function (see below).



Simulation of a gamma subordinator.

Before we go further into the probabilistic properties of subordinators we'll make a quick diversion into analysis.
Let $f \in C^{\infty}((0,\infty))$. We say it is *completely monotone* if $(-1)^n f^{(n)} \geq 0$ for all $n \in \mathbb{N}$, and a *Bernstein function* if $f \geq 0$ and $(-1)^n f^{(n)} \leq 0$ for all $n \in \mathbb{N}$.

Theorem

1. *f is a Bernstein function if and only if the mapping $x \rightarrow e^{-tf(x)}$ is completely monotone for all $t \geq 0$.*
2. *f is a Bernstein function if and only if it has the representation*

$$f(x) = a + bx + \int_0^{\infty}(1 - e^{-yx})\lambda(dy),$$

for all $x > 0$ where $a, b \geq 0$ and $\int_0^{\infty}(y \wedge 1)\lambda(dy) < \infty$.

3. *g is completely monotone if and only if there exists a measure μ on $[0,\infty)$ for which*

$$g(x) = \int_0^{\infty} e^{-xy}\mu(dy).$$

To interpret this theorem, first consider the case $a = 0$. In this case, if we compare the statement of Theorem 4 with equation (2.5), we see that there is a one to one correspondence between Bernstein functions for which $\lim_{x \to 0} f(x) = 0$ and Laplace exponents of subordinators. The Laplace transforms of the laws of subordinators are always completely monotone functions and a subclass of all possible measures μ appearing in Theorem 4 (3) is given by all possible laws $p_{T(t)}$ associated to subordinators. A general Bernstein function with $a > 0$ can be given a probabilistic interpretation by means of "killing".

One of the most important probabilistic applications of subordinators is to "time change". Let X be an arbitrary Lévy process and let T be a subordinator defined on the same probability space as X such that X and T are independent. We define a new stochastic process $Z = (Z(t), t \geq 0)$ by the prescription

$$Z(t) = X(T(t)),$$

for each $t \geq 0$ so that for each $\omega \in \Omega, Z(t)(\omega) = X(T(t)(\omega))(\omega)$. The key result is then the following.

Theorem

Z is a Lévy process.

We compute the Lévy symbol of the subordinated process Z.

Theorem

$$\eta_Z = -\psi_T \circ (-\eta_X).$$

Proof. For each $u \in \mathbb{R}^d, t \geq 0$,

$$\begin{aligned} e^{i\eta_{Z(t)}(u)} &= \mathbb{E}(e^{i(u,Z(t))}) \\ &= \mathbb{E}(e^{i(u,X(T(t)))}) \\ &= \int \mathbb{E}(e^{i(u,X(s))}) p_{T(t)}(ds) \\ &= \int e^{s\eta_X(u)} p_{T(t)}(ds) \\ &= \mathbb{E}(e^{-(-\eta_X(u))T(t)}) \\ &= e^{-t\psi_T(-\eta_X(u))}. \qquad \square \end{aligned}$$

Example : From Brownian Motion to 2α-stable Processes

Let T be an α-stable subordinator (with $0 < \alpha < 1$) and X be a d-dimensional Brownian motion with covariance $A = 2I$, which is independent of T. Then for each $s \geq 0, u \in \mathbb{R}^d, \psi_T(s) = s^\alpha$ and $\eta_X(u) = -|u|^2$, and hence $\eta_Z(u) = -|u|^{2\alpha}$, i.e. Z is a rotationally invariant 2α-stable process.
In particular, if $d = 1$ and T is the Lévy subordinator, then Z is the *Cauchy process*, so each $Z(t)$ has a symmetric Cauchy distribution with parameters $\mu = 0$ and $\sigma = 1$. It is interesting to observe from (2.6) that Z is constructed from two independent standard Brownian motions.

Examples of subordinated processes have recently found useful applications in mathematical finance. We briefly mention two interesting cases:-

(i) **The Variance Gamma Process**

In this case $Z(t) = B(T(t))$, for each $t \geq 0$, where B is a standard Brownian motion and T is an independent gamma subordinator. The name derives from the fact that, in a formal sense, each $Z(t)$ arises by replacing the variance of a normal random variable by a gamma random variable. Using Theorem 6, a simple calculation yields

$$\Phi_{Z(t)}(u) = \left(1 + \frac{u^2}{2b}\right)^{-at},$$

for each $t \geq 0, u \in \mathbb{R}$, where a and b are the usual parameters which determine the gamma process. It is an easy exercise in manipulating characteristic functions to compute the alternative representation:

$$Z(t) = G(t) - L(t),$$

where G and L are independent gamma subordinators each with parameters $\sqrt{2b}$ and a. This yields a nice financial representation of Z as a difference of independent "gains" and "losses". From this representation, we can compute that Z has a Lévy density

$$g_\nu(x) = \frac{a}{|x|}(e^{\sqrt{2b}x}\mathbf{1}_{(-\infty,0)}(x) + e^{-\sqrt{2b}x}\mathbf{1}_{(0,\infty)}(x)).$$

The *CGMY processes* are a generalisation of the variance-gamma processes due to Carr, Geman, Madan and Yor. They are characterised by their Lévy density:

$$g_\nu(x) = \frac{a}{|x|^{1+\alpha}}(e^{b_1 x}\mathbf{1}_{(-\infty,0)}(x) + e^{-b_2 x}\mathbf{1}_{(0,\infty)}(x)),$$

where $a > 0, 0 \leq \alpha < 2$ and $b_1, b_2 \geq 0$. We obtain stable Lévy processes when $b_1 = b_2 = 0$. They can also be obatined by subordinating Brownian motion with drift. The CGMY processes are a subclass of the *tempered stable processes*. Note how the exponential dampens the effects of large jumps.

(ii) **The Normal Inverse Gaussian Process**

In this case $Z(t) = C(T(t)) + \mu t$ for each $t \geq 0$ where each $C(t) = B(t) + \beta t$, with $\beta \in \mathbb{R}$. Here T is an inverse Gaussian subordinator, which is independent of B, and in which we write the parameter $\gamma = \sqrt{\alpha^2 - \beta^2}$, where $\alpha \in \mathbb{R}$ with $\alpha^2 \geq \beta^2$. Z depends on four parameters and has characteristic function

$$\Phi_{Z(t)}(\alpha, \beta, \delta, \mu)(u) = \exp\{\delta t(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}) + i\mu t u\}$$

for each $u \in \mathbb{R}, t \geq 0$. Here $\delta > 0$ is as in (2.7).
Each $Z(t)$ has a density given by

$$f_{Z(t)}(x) = C(\alpha, \beta, \delta, \mu; t) q\left(\frac{x - \mu t}{\delta t}\right)^{-1} K_1\left(\delta t \alpha q\left(\frac{x - \mu t}{\delta t}\right)\right) e^{\beta x},$$

for each $x \in \mathbb{R}$, where
$q(x) = \sqrt{1 + x^2}$, $C(\alpha, \beta, \delta, \mu; t) = \pi^{-1}\alpha e^{\delta t\sqrt{\alpha^2 - \beta^2} - \beta\mu t}$ and K_1 is a Bessel function of the third kind.