Lectures on Lévy Processes and Stochastic Calculus, Braunschweig, Lecture 5: Lecture 5 The Ornstein-Uhlenbeck Process

David Applebaum

Probability and Statistics Department, University of Sheffield, UK

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Historical Origins

This process was first introduced by Ornstein and Uhlenbeck in the 1930s as a more accurate model of the physical phenomenon of Brownian motion than the Einstein-Smoluchowski-Wiener process. They argued that

Brownian motion = viscous drag of fluid + random molecular bombardment.

Let $v(t)$ be the velocity at time t of a particle of mass m executing Brownian motion. By Newton's second law of motion, the total force acting on the particle at time t is $F(t) = m\frac{dv(t)}{dt}$.We then have

$$m\frac{dv(t)}{dt} = -\underbrace{mkv(t)}_{\text{viscous drag}} + \underbrace{m\sigma\frac{dB(t)}{dt}}_{\text{molecular bombardment}},$$

where $k, \sigma > 0$.

Of course, $\frac{dB(t)}{dt}$ doesn't exist, but this is a "physicist's argument". If we cancel the ms and multiply both sides by dt then we get a legitimate SDE - the *Langevin equation*

$$dv(t) = -kv(t)dt + \sigma dB(t) \qquad (0.1)$$

Using the integrating factor e^{kt} we can then easily check that the unique solution to this equation is the *Ornstein-Uhlenbeck process* $(v(t), t \geq 0)$ where

$$v(t) = e^{-kt}v(0) + \int_0^t e^{-k(t-s)}dB(s).$$

We are interested in Lévy processes so replace B by a d-dimensional Lévy process X and k by a $d \times d$ matrix K. Our Langevin equation is

$$dY(t) = -KY(t)dt + dX(t) \qquad (0.2)$$

and its unique solution is

$$Y(t) = e^{-tK}Y_0 + \int_0^t e^{-(t-s)K}dX(s), \qquad (0.3)$$

where $Y_0 := Y(0)$ is a fixed $\mathcal{F}_0$ measurable random variables. We still call the process Y an *Ornstein-Uhlenbeck* or OU process. urthermore

- Y has càdlàg paths.
- Y is a Markov process.

The process X is sometimes called the *background driving Lévy process* or BDLP.

We get a Markov semigroup on $B_b(\mathbb{R}^d)$ called a *Mehler semigroup*:

$$\begin{aligned} T_t f(x) &= \mathbb{E}(f(Y(t))|Y_0 = x) \\ &= \int_{\mathbb{R}^d} f(e^{-tK}x + y)\rho_t(dy) \end{aligned} \tag{0.4}$$

where ρ_t is the law of the stochastic integral $\int_0^t e^{-sK}dX(s) \stackrel{d}{=} \int_0^t e^{-(t-s)K}dX(s)$.
This generalises the classical Mehler formula ($X(t) = B(t), K = kI$)

$$T_t f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f\left(e^{-kt}x + \sqrt{\frac{1 - e^{-2kt}}{2k}}y\right) e^{-\frac{y^2}{2}} dy.$$

In fact $(T_t, t \geq 0)$ satisfies the Feller property: $T_t(C_0(\mathbb{R}^d)) \subseteq C_0(\mathbb{R}^d)$.
We also have the *skew-convolution semigroup* property:

$$\rho_{s+t} = \rho_s^K * \rho_t,$$

where $\rho_s^K(B) = \rho_s(e^{tK}B)$. Another terminology for this is *measure-valued cocycle*.

We get nicer probabilistic properties of our solution if we make the following

Assumption K is strictly positive definite.

OU processes solve simple linear SDEs. They are important in applications such as volatility modelling, Lévy driven CARMA processes, branching processes with immigration.
In infinite dimensions they solve the simplest linear SPDE with additive noise. To develop this theme, let H and K be separable Hilbert spaces and $(S(t), t \geq 0)$ be a C_0-semigroup on H with infinitesimal generator J. Let X be a Lévy process on K and $C \in \mathcal{L}(K, H)$.

We have the SPDE

$$dY(t) = JY(t) + CdX(t),$$

whose unique solution is

$$Y(t) = S(t)Y_0 + \underbrace{\int_0^t S(t-s)CdX(s)}_{\text{stochastic convolution}},$$

and the generalised Mehler semigroup is

$$T_t f(x) = \int_{\mathbb{R}^d} f(S(t)x + y)\rho_t(dy).$$

From now on we will work in finite dimensions and assume the strict positive-definiteness of K.

Additive Processes and Wiener-Lévy Integrals

The study of O-U processes focusses attention on Wiener-Lévy integrals $I_f(t) := \int_0^t f(s)dX(s)$. For simplicity we assume that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuous.

Recall that $Z = (Z(t), t \geq 0)$ is an *additive process* if $Z(0) = 0$ (a.s.), Z has independent increments and is stochastically continuous. It follows that each $Z(t)$ is infinitely divisible.

Theorem

$(I_f(t), t \geq 0)$ *is an additive process.*

Proof. (sketch) Independent increments follows from the fact that for $r \leq s \leq t$

$I_f(s) - I_f(r) = \int_r^s f(u)dX(u)$ is $\sigma\{X(b) - X(s); r \leq a < b \leq s\}$ - measurable,

$I_f(t) - I_f(s) = \int_s^t f(u)dX(u)$ is $\sigma\{X(d) - X(c); s \leq c < d \leq t\}$ - measurable, □

Theorem

If X has Lévy symbol η then for each $t \geq 0, u \in \mathbb{R}^d$,

$$\mathbb{E}(e^{i(u, I_f(t))}) = \exp\left\{\int_0^t \eta(f(s)^T u)\right\}.$$

Proof. (sketch) Define $M_f(t) = \exp\left\{i\left(u, \int_0^t f(s)dX(s)\right)\right\}$ and use Itô's formula to show that

$$\begin{aligned} M_f(t) &= 1 + i\left(u, \int_0^t M_f(s-)f(s)dB(s)\right) \\ &+ \int_0^t \int_{\mathbb{R}^d - \{0\}} M_f(s-)(e^{i(u, f(s)x)} - 1)\tilde{N}(ds, dx) + \int_0^t M_f(s-)\eta(f(s)^T u)ds. \end{aligned}$$

Now take expectations of both sides to get

$$\mathbb{E}(M_f(t)) = 1 + \int_0^t \mathbb{E}(M_f(s))\eta(f(s)^T u)ds,$$

and the result follows. □

If X has characteristics (b, A, ν), it follows that $I_f(t)$ has characteristics (b_t^f, A_t^f, ν_t^f) where

$$b_t^f = \int_0^t f(s)bds + \int_0 \int_{\mathbb{R}^d - \{0\}} f(s)x(\mathbf{1}_{\hat{B}}(x) - \mathbf{1}_{\hat{B}}(f(s)x))\nu(dx)ds,$$

$$A_t^f = \int_0^t f(s)^T A f(s)ds,$$

$$\nu_t^f(B) = \int_0^t \nu(f(s)^{-1}(B)).$$

It follows that every OU process Y conditioned on $Y_0 = y$ is an additive process. It will have characteristics as above with $f(s) = e^{-sK}$ and b_t^f translated by $e^{-tK}y$.

Invariant Measures, Stationary Processes, Ergodicity: General Theory

We want to investigate invariant measures and stationary solutions for OU processes. First a little general theory.
First let $(T_t, t \geq 0)$ be a general Markov semigroup with transition probabilities $p_t(x, B) = T_t \mathbf{1}_B(x)$ so that $T_t f(x) = \int_{\mathbb{R}^d} f(y) p_t(x, dy)$ for $f \in B_b(\mathbb{R}^d)$. We say that a probability measure μ is an *invariant measure* for the semigroup if for all $t \geq 0, f \in B_b(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} T_t f(x) \mu(dx) = \int_{\mathbb{R}^d} f(x) \mu(dx) \quad (0.5)$$

Equivalently for all Borel sets B

$$\int_{\mathbb{R}^d} p_t(x, B) \mu(dx) = \mu(B). \quad (0.6)$$

To see that (0.5) $\Rightarrow$ (0.6) rewrite as

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) p_t(x, dy) \mu(dx) = \int_{\mathbb{R}^d} f(x) \mu(dx),$$

and put $f = \mathbf{1}_B$. For the converse - approximate f by simple functions and take limits.

e.g. A Lévy process doesn't have an invariant probability measure but Lebesgue measure is invariant in the sense that for $f \in L^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} T_t f(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x + y) p_t(dy) dx = \int_{\mathbb{R}^d} f(x) dx.$$

A process $Z = (Z(t), t \geq 0)$ is (strictly) *stationary* if for all $n \in \mathbb{N}, t_1, \ldots, t_n, h \in \mathbb{R}^+$,

$$(Z(t_1), \ldots, Z(t_n)) \stackrel{d}{=} (Z(t_1 + h), \ldots, Z(t_n + h))$$

Theorem

A Markov process Z wherein μ is the law of $Z(0)$ is stationary if and only if μ is an invariant measure.

Proof. If the process is stationary then μ is invariant since

$$\mu(B) = P(Z(0) \in B) = P(Z(t) \in B) = \int_{\mathbb{R}^d} p_t(x, B) \mu(dx).$$

For the converse, its sufficient to prove that $\mathbb{E}(f_1(Z(t_1 + h)) \cdots f_n(Z(t_n + h))))$ is independent of h for all $f_1, \ldots f_n \in B_b(\mathbb{R}^d)$. Proof is by induction. Case $n = 1$. Its enough to show

$$\begin{aligned} \mathbb{E}(f(Z(t)) &= \mathbb{E}(\mathbb{E}(f(Z(t)|\mathcal{F}_0))) \\ &= \mathbb{E}(T_t f(Z(0))) \\ &= \int_{\mathbb{R}^d} (T_t f(x)) \mu(dx) \\ &= \int_{\mathbb{R}^d} f(x) dx = \mathbb{E}(f(Z(0))). \end{aligned}$$

In general use

$$\begin{aligned} & \mathbb{E}(f_1(Z(t_1+h))\cdots f_n(Z(t_n+h))) \\ = & \mathbb{E}(f_1(Z(t_1+h))\cdots \mathbb{E}(f_n(Z(t_n+h))|\mathcal{F}_{t_{n-1}+h})) \\ = & \mathbb{E}(f_1(Z(t_1+h)\cdots T_{t_n-t_{n-1}}f_n(Z(t_{n-1}+h)))). \end{aligned}$$

□

Let μ be an invariant probability measure for a Markov semigroup $(T_t, t \geq 0)$. μ is *ergodic* if

$$T_t \mathbf{1}_B = \mathbf{1}_B (\mu \text{ a.s.}) \Rightarrow \mu(B) = 0 \text{ or } \mu(B) = 1.$$

If μ is ergodic then "time averages" = "space averages" for the corresponding stationary Markov process, i.e.

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(Z(s))ds = \int_{\mathbb{R}^d} f(x)\mu(dx) \text{ a.s.}$$

Fact: The invariant measures form a convex set and the ergodic measures are the extreme points of this set.

It follows that if an invariant measure is *unique* then it is ergodic.

The Self-Decomposable Connection

Recall that a random variable Z is *self-decomposable* if for each $0 < a < 1$ there exists a random variable W_a that is independent of Z such that

$$Z \stackrel{d}{=} aZ + W_a$$

or equivalently $\rho_Z = \rho_Z^a * \rho_{W_a}$, where $\rho_Z^a(B) = \rho(a^{-1}B)$.
Now suppose that Y is a stationary Ornstein-Uhlenbeck process on $\mathbb{R}$. Then Y_0 is self decomposable with $a = e^{-kt}$ and $W_{a(t)} = \int_0^t e^{-ks}dX(s)$ since

$$Y(t) = e^{-kt}Y_0 + \int_0^t e^{-(t-s)K}dX(s)$$

and by stationary increments of the process X

$$Y(t) \stackrel{d}{=} Y_0 \text{ and } \int_0^t e^{-k(t-s)}dX(s) \stackrel{d}{=} \int_0^t e^{-ks}dX(s)$$

$$\Rightarrow Y_0 \stackrel{d}{=} e^{-kt}Y_0 + W_{a(t)}.$$

Now suppose that μ is self-decomposable - more precisely that

$$\mu = \mu^{e^{kt}} * \rho_t,$$

where ρ_t is the law of $W_{a(t)}$. Then

$$\begin{aligned}\int_{\mathbb{R}} T_t f(x)\mu(dx) &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(e^{-kt}x + y)\rho_t(dy)\mu(dx)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x+y)\rho_t(dy)\mu^{e^{kt}}(dx)\\ &= \int_{\mathbb{R}} f(x)(\mu^{e^{kt}} * \rho_t)(dx)\\ &= \int_{\mathbb{R}} f(x)\mu(dx).\end{aligned}$$

So μ is an invariant measure.

So we have shown that:

Theorem

The following are equivalent for the O-U process Y.

- *Y is stationary.*
- *The law of $Y(0)$ is an invariant measure.*
- *The law of $Y(0)$ is self-decomposable (with $W_{a(t)} = \int_0^t e^{-ks} dX(s)$).*

We seek some condition on the Lévy process X which ensures that Y is stationary.

Fact: If $Y_\infty := \int_0^\infty e^{-ks} dX(s)$ exists in distribution then it is self-decomposable.

To see this observe that (using stationary increments of X)

$$\begin{aligned}\int_0^\infty e^{-ks} dX(s) &= \int_t^\infty e^{-ks} dX(s) + \int_0^t e^{-ks} dX(s)\\ &\stackrel{d}{=} \int_0^\infty e^{-k(t+s)} dX(s) + \int_0^t e^{-ks} dX(s)\\ &= e^{-kt}\int_0^\infty e^{-ks} dX(s) + \int_0^t e^{-ks} dX(s)\end{aligned}$$

When does $\lim_{t\to\infty} \int_0^t e^{-ks} dX(s)$ exist in distribution? Use the Lévy-Itô decomposition.

$$X(t) = bt + M(t) + \int_{|x|\geq 1} xN(t, dx).$$

It is not difficult to see that $\lim_{t\to\infty} \int_0^t e^{-ks} dM(s)$ exists in L^2-sense.

Fact: $\lim_{t\to\infty} \int_0^t \int_{|x|\geq 1} e^{-ks} xN(ds, dx)$ exists in distribution if and only if $\int_{|x|\geq 1} \log(1 + |x|)\nu(dx) < \infty$.

To prove this you need

1. If $(\xi_n, n \in \mathbb{N})$ are i.i.d. then $\sum_{n=1}^{\infty} c^n \xi_n$ converges a.s. $(0 < c < 1)$ if and only if $\mathbb{E}(\log(1 + |\xi_1|)) < \infty$.
2. $$\int_0^n \int_{|x|\geq 1} e^{-ks} x N(ds, dx) \stackrel{d}{=} \sum_{j=0}^{n-1} e^{-kj} M_j$$
   where $M_j := \int_j^{j+1} \int_{|x|\geq 1} e^{-k(s-j)} x N(ds, dx)$. Note that $(M_j, j \in \mathbb{N})$ are i.i.d.

In this case, Y has characteristics $(b_\infty^f, A_\infty^f, \nu_\infty^f)$.
e.g. Brownian motion case. $X(t) = B(t)$. $\mu \sim N\left(0, \frac{1}{2k}\right)$.

In fact, if an invariant measure μ exists then it is unique. For suppose that Y is stationary, then

$$Y(0) \stackrel{d}{=} e^{-kt} Y(0) + \int_0^t e^{-ks} dX(s).$$

Now let ρ be the law of $Y(0)$ and $\Phi_\rho(u) := \int_{\mathbb{R}} e^{iuy} \rho(dy)$. Then for all $u \in \mathbb{R}$, by independence

$$\Phi_\rho(u) = \Phi_\rho(e^{-kt}u) \exp\left\{-\int_0^t \eta(e^{-ks}u) ds\right\}.$$

Take limits as $t \to \infty$ to get

$$\Phi_\rho(u) = \exp\left\{-\int_0^\infty \eta(e^{-ks}u) ds\right\}.$$

So ρ is the law of Y_∞.

Example: Let $(X(t), t \geq 0)$ be a compound Poisson process $X(t) = \sum_{i=1}^{N(t)} W_i$ where the W_is are i.i.d. exponential with common density $f_W(x) = ae^{-ax} \mathbf{1}_{x>0}$. Then

$$\eta(u) = \lambda a \int_0^\infty (e^{iux} - 1) e^{-ax} dx = \frac{\lambda a}{a - iu}.$$

You can check that $\Phi_\rho(u) = (1 - ia^{-1}u)^{-\lambda}$ as so ρ has a gamma(c, λ) distribution.

In fact - it is possible to go further. Given any self-decomposable distribution μ there exists a stationary Ornstein-Uhlenbeck process Y such that the law of $Y(0)$ is μ. Let's sketch the proof of this - due to Jurek and Vervaat (1983). Let X be a self-decomposable random variable with distribution μ. Then for each $t \geq 0$

$$X \stackrel{d}{=} e^{-t} X + X_t,$$

where X and X_t are independent.
The key step is the observation that we can construct an additive process $(Z(t), t \geq 0)$ such that

$$Z(t) \stackrel{d}{=} X_t \text{ and } Z(t+h) - Z(t) \stackrel{d}{=} e^{-t} X_h.$$

This follows by Kolmogorov's theorem since

$$\begin{aligned} X &\stackrel{d}{=} e^{-(t+h)}X + X_{t+h} \\ &\stackrel{d}{=} e^{-t}(e^{-h}X + X_h) + X_t \\ \Rightarrow X_{t+h} &\stackrel{d}{=} e^{-t}X_h + X_t. \end{aligned}$$

It follows that $Y(t) = \int_0^t e^s dZ(s)$ also has independent increments. But Y is a Lévy process since

$$\begin{aligned} Y(t+h) - Y(t) &= \int_t^{t+h} e^s dZ(s) \\ &= \int_0^h e^s e^t dZ(s+t) \\ &\stackrel{d}{=} \int_0^h e^s e^t e^{-t} dZ(s) = Y(h) \end{aligned}$$

We then find that $Z(t) = \int_0^t e^{-s} dY(s)$ and so

$$X \stackrel{d}{=} e^{-t}X + \int_0^t e^{-s}dY(s) \stackrel{d}{=} e^{-t}X + \int_0^t e^{-(t-s)}dY(s),$$

using stationary increments of Y which is extended to an Lévy process on the whole of $\mathbb{R}$.

In the 1990s, Sato showed that μ is self-decomposable if and only if it is the law of $W(1)$ where $(W(t), t \geq 0)$ is a self-similar additive process. Recall W self-similar (index H) means for all $c \geq 0$

$$W(ct) \stackrel{d}{=} c^H W(t).$$

So we can embed selfdecomposable distributions into stationary OU processes and self-similar additive processes. Is there a connection?

To understand the connection between the two "embeddings" of μ we need the

Lamperti Transform. There is a one-to-one correspondence between self-similar processes $(W(t), t \geq 0)$ and stationary processes $(Z(t), t \geq 0)$ given by

$$W(t) = t^H Z(\log(t)) \text{ or equivalently } Z(t) = e^{-tH}W(e^t).$$

Indeed if W self-similar

$$\begin{aligned} Z(t+h) &= e^{-(t+h)H}W(e^{t+h}) \\ &\stackrel{d}{=} e^{-tH}e^{-hH}e^{hH}W(e^t) \\ &= Z(t). \end{aligned}$$

The next step is due to Jeanblanc, Pitman, Yor (SPA 100, 223(2002))

Start with a self-similar additive process $(W(t), t \geq 0)$. Then we know that $W(1)$ is self-decomposable. There exist two independent, identically distributed Lévy processes $(X_t^-, t \geq 0)$ and $(X_t^+, t \geq 0)$ such that

$$X_t^- = \int_{e^{-t}}^1 \frac{dW(r)}{r^H}, X_t^+ = \int_1^{e^t} \frac{dW(r)}{r^H}.$$

Let $(Z(t), t \geq 0)$ be the stationary Lamperti transform of W. Then it is an Ornstein-Uhlenbeck process and

$$Z(t) = e^{-tH}W(1) + \int_0^t e^{-(t+s)H}dX_s^+,$$

$$Z(-t) = e^{-tH}W(1) - \int_0^t e^{-(t+s)H}dX_s^-.$$

In the last part of the lecture we'll briefly look at some recent developments.

Densities of the OU Process

We've seen that each $Y(t)$ is infinitely divisible so if the Lévy process $X(t)$ has a Gaussian component then so does $Y(t)$ in which case it has a density by Fourier inversion.
More generally, Priola and Zabczyk (BLMS, 41, 41,(2009)) study

$$dY(t) = AY(t)dt + BdX(t),$$

where each $Y(t)$ is $\mathbb{R}^d$-valued but $X(t)$ is $\mathbb{R}^n$-valued ($n \geq d$). So A is an $n \times n$ matrix and B is an $n \times d$ matrix.

Assume

1. $\text{Rank}[B, AB, \ldots, A^{n-1}B] = n$,
   where $[B, AB, \ldots, A^{n-1}B]$ is the matrix of the linear mapping from $\mathbb{R}^{nd}$ to $\mathbb{R}^n$ given by

$$(u_0, u_1, \ldots, u_{n-1}) \to Bu_0 + ABu_1 + \ldots + A^{n-1}Bu_{n-1}.$$

2. The restriction of the Lévy measure ν to $B_r(0)$ has a density for some $r > 0$.

Then $Y(t)$ has a density for $t > 0$.

Application - Volatility Modelling

Consider the Black-Scholes model for a stock price

$$S(t) = S(0)\exp\{\mu t + \sigma B(t)\},$$

where $\mu \in \mathbb{R}$ is stock drift and $\sigma > 0$ is volatility. By Itô's formula

$$dS(t) = \sigma S(t)dB(t) + S(t)\left(\mu + \frac{1}{2}\sigma^2\right)dt.$$

In *stochastic volatility models* the parameter σ^2 is replaced by a stochastic process $(\sigma^2(t), t \geq 0)$.

Barndorff-Nielsen and Shephard (JRSS B 63, 167 (2001)) proposed the OU model

$$d\sigma^2(t) = -\lambda\sigma^2(t) + dX(\lambda t),$$

where $\lambda > 0$ and X is a subordinator. Then $\sigma^2(t) > 0$ (a.s.) since

$$\begin{aligned}\sigma^2(t) &= e^{-\lambda t}\sigma^2(0) + \int_0^t e^{-\lambda(t-s)}dX(\lambda t)\\ &= e^{-\lambda t}\left\{\sigma^2(0) + \sum_{0 \leq u \leq t} e^{-\lambda u}\Delta X(\lambda u)\right\}.\end{aligned}$$

Assume that $\int_1^\infty \log(1+x)\nu(dx) < \infty$. Then there is a unique invariant measure μ which is self-decomposable and has characteristic function

$$\widehat{\mu}(u) = \exp\left\{\int_0^\infty (e^{iux} - 1)\frac{k(x)}{x}dx\right\},$$

where k is decreasing.

Problem. Based on discrete-time observations $\sigma^2(0), \sigma^2(\Delta), \sigma^2((N-1)\Delta)$ find estimates of the parameter λ and k. For a non-parametric approach - see Jongbloed et. al. Bernoulli, 11, 759 (2005).

Generalised Ornstein-Uhlenbeck Processes

Let $X = (X_1, X_2)$ be a Lévy process on $\mathbb{R}^2$. Then each X_i is a real-valued Lévy process. Let Y_0 be independent of X. The *generalised Ornstein-Uhlenbeck process* is

$$Y(t) = e^{-X_1(t)}\left(Y_0 + \int_0^t e^{-X_1(s)}dX_2(s)\right).$$

The usual OU process is obtained by taking $X_1(t) = \lambda t$ $(\lambda > 0)$. Necessary and sufficient conditions for stationarity solutions were found by Lindner and Maller (SPA 115, 1701 (2005)). Almost sure convergence of $\int_0^t e^{-X_1(s)}dX_2(s)$ as $t \to \infty$ is a sufficient condition but the general story is more complicated.

In fact a necessary and sufficient condition for stationary solutions is the almost sure convergence of $\int_0^t e^{-X_1(s)}dL(s)$ as $t \to \infty$ where the one-dimensional Lévy process $(L(t), t \geq 0)$ is defined by

$$L(t) := X_2(t) + \sum_{0 \leq s \leq t}(e^{-\Delta X_1(s)} - 1)\Delta X_2(s) - t\tilde{A}_{1,2},$$

where $\tilde{A}_{1,2}$ is the off-diagonal entry of the covariance matrix of the Gaussian component of the bivariate Lévy process (X_1, L). For further work on generalised O-U processes see Lindner and Sato (AP 37, 250 (2009)).

References and Further Reading

These lectures have been broadly based on my recent book:

D.Applebaum *Lévy Processes and Stochastic Calculus*, Cambridge University Press (second edition) (2009)

and from an earlier course of lectures partly derived from it, which have been separately published as

D.Applebaum, Lévy processes in Euclidean spaces and groups in *Quantum Independent Increment Processes I: From Classical Probability to Quantum Stochastic Calculus, Springer Lecture Notes in Mathematics* , Vol. **1865** M Schurmann, U. Franz (Eds.) 1-99,(2005)

A comprehensive account of the structure and properties of Lévy processes is:

K-I.Sato, *Lévy Processes and Infinite Divisibility*, Cambridge University Press (1999)

A shorter account, from the point of view of the French school, which concentrates on fluctuation theory and potential theory aspects is

J.Bertoin, *Lévy Processes*, Cambridge University Press (1996)

From this point of view, you should also look at
A.Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer-Verlag (2006)

For an insight into the wide range of both theoretical and applied recent work wherein Lévy processes play a role, consult
O.E.Barndorff-Nielsen,T.Mikosch, S.Resnick (eds), *Lévy Processes: Theory and Applications*, Birkhäuser, Basel (2001)

For stochastic calculus with jumps, the authoritative treatise is
P.Protter, *Stochastic Integration and Differential Equations* (second edition), Springer-Verlag, Berlin Heidelberg (2003)

For financial modelling I recommend:

R.Cont, P.Tankov, *Financial Modelling with Jump Processes*, Chapman and Hall/CRC (2004)

which is extremely comprehensive and also contains a lot of valuable background material on Lévy processes.

W.Schoutens, *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley (2003)

is shorter and aimed at a wider audience than mathematicians and statisticians.