University of Sheffield

School of Mathematics & and Statistics

Analysis

MAS221

Semester 1 2016-17

Chapter 1

Numbers, Inequalities, Bounds, and Completeness

1.1 What is Analysis?

Broadly speaking, *analysis* is the theory of the *limit*. It serves as a foundation for the calculus, and also enables us to give a mathematically rigorous description of the real number line. Like algebra, geometry and topology, it is one of the central themes of pure mathematics and is still being actively developed by research mathematicians. It is also a vital tool in applications to e.g. quantum theory, probability theory, finance and economics ...

In this course, we will study *real analysis*, so our focus is on real numbers and real-valued functions. In later years, you may choose to study *complex analysis* (MAS332), *functional analysis* (MAS436) – where analysis meets linear algebra), or other parts of analysis that build on the work we'll do here, such as *metric spaces* (MAS331), and *measure theory* (MAS350/451).

The modern theory of the *calculus*, which is a mathematical theory of change and motion, was to a large extent, discovered independently by two great scientists: Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), but despite its amazing success in applications, it was not a logically coherent theory at the time that it first burst on the scene; more a collection of useful mathematical methods. You all know the definition of the derivative at a point x of a "sufficiently nice function" f:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x};$$

but how we know that df/dx exists? What does $\lim_{\Delta x \to 0}$ really mean? What is the "infinitesimal quantity Δx, and how can it "tend to zero"? In the eighteenth century, the calculus was fiercely criticised by the British philosopher

George Berkeley, who called infinitesimals like Δx: "the ghost of departed quantities".

Many mathematicians rose to Berkeley's challenge and tried to give a satisfactory foundation to the calculus, but it wasn't until the nineteenth century that modern analysis was born, and a logically rigorous meaning was given to the concept of the limit. This was mainly the work of three great mathematicians: Augustus Louis Cauchy (1789–1857), Bernhard Bolzano (1781–1848), and Karl Weierstrass (1815–97).

One of the key aims of this course is to understand the definition of the limit, discover its properties, learn how to use these, and also give a more thorough foundation to the real number line, and to the calculus.

In Semester 1, we will study sets of real numbers, sequences and their limits, limits of functions, functions that are continuous on the whole real number line, and also on finite intervals, differentiation as a limit, and Taylor series. Along the way, we will prove a lot of new theorems, many of which are also very useful in applications.

In Semester 2, you will study the convergence of infinite series, integration, the analytic foundations of calculus in higher dimension, and a new important concept called *uniform convergence.*

Analysis is intellectually beautiful. We hope that you enjoy it !!

1.2 Irrational Numbers

We use the following standard notation for sets of numbers:

$$\text{The natural numbers } \mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$
$$\text{The non-negative integers } \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.$$
$$\text{The integers } \mathbb{Z} = \{, \ldots, -5, -4, -3, -2, 1, 0, 1, 2, 3, \ldots\}.$$
$$\text{The rational numbers } \mathbb{Q} = \{p/q; p \in \mathbb{Z}, q \in \mathbb{N}\}.$$
$$\text{The real numbers } \mathbb{R}.$$

Our intuitive understanding of $\mathbb{R}$ is the set of all points on an infinite straight line of zero width. We write

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c,$$

where $\mathbb{Q}^c$ is the set of all *irrational numbers*, i.e. real numbers that cannot be written as ratios of whole numbers.

You are aware that $\sqrt{2}, e, \pi$ are irrational numbers. In MAS114, you showed that the sum of a rational and an irrational number is irrational. It isn't hard to prove that the product of a non-zero rational and an irrational number is irrational. To increase our supply of irrational numbers, we will prove that $\sqrt{p}$ is irrational for every prime number p, so $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}$, etc are irrational. Since there are infinitely many prime numbers, this immediately tells us that there are infinitely many irrational numbers.

We will need the *prime factorisation theorem* which was proved in MAS114. This states that every natural number is uniquely given as a product of primes. So if $n \in \mathbb{N}$, we may write

$$n = 2^{m_1}3^{m_2}5^{m_3}7^{m_4}\cdots p^{m_N}, \tag{1.2.1}$$

where p is the largest prime that we need, and $m_1, m_2, \ldots, m_N \in \mathbb{Z}_+$, e.g. $7920 = 2^4.3^2.5.11$, so in this case, $p = 11$, $m_1 = 4, m_2 = 2, m_3 = 1, m_4 = 0, m_5 = 1$. The next result presupposes that square roots exist, and we will revisit this later.

Theorem 1.2.1. *If p is a prime number then $\sqrt{p}$ is irrational.*

Proof. Assume that $\sqrt{p}$ is rational and seek a contradiction. Write $\sqrt{p} = \dfrac{a}{b}$ and then square both sides to get $p = \dfrac{a^2}{b^2}$ so that $a^2 = b^2p$.
Write a in terms of its prime decomposition as $a = 2^{m_1}3^{m_2}5^{m_3}\cdots q^{m_N}$. If we square this we get

$$a^2 = 2^{2m_1}3^{2m_2}5^{2m_3}\cdots q^{2m_N}.$$

Now we do the same for b. We write its prime factorisation as $b = 2^{n_1}3^{n_2}5^{n_3}\cdots r^{n_M}$, and square this to get

$$b^2 = 2^{2n_1}3^{2n_2}5^{2n_3}\cdots r^{2n_M}.$$

Return to the equation $a^2 = b^2p$, and substitute in the prime factorisations for a^2 and b^2. We get

$$2^{2m_1}3^{2m_2}5^{2m_3}\cdots q^{2m_N} = 2^{2n_1}3^{2n_2}5^{2n_3}\cdots r^{2n_M}p.$$

Now if the number p doesn't appear on the left hand side we already have a contradiction, so lets assume that it does, and that it is one of the numbers $2, 3, 5, \ldots, q$. Each of these prime numbers appears an *even* number of times on the left hand side. On the right hand side, either p is not one of the numbers $2, 3, \ldots, r$, in which case it only appears *once* altogether, or it is in that list of numbers, in which case the extra multiplication by p means that it appears an *odd* number of times. Either way we have a contradiction, and so we conclude that $\sqrt{p}$ cannot be a rational number. □

The result just proved has a very interesting consequence:

Theorem 1.2.2. *Given any two rational numbers a and b, with $a < b$, we can find infinitely many irrational numbers q such that*

$$a < q < b.$$

Proof. Let p be a prime number. Since $p > 1$ then $\sqrt{p} > 1$, and so $\dfrac{1}{\sqrt{p}} < 1$. Now define $q = a + \frac{1}{\sqrt{p}}(b-a)$. Then q is irrational, and since there are infinitely many prime numbers, it follows that there are infinitely many numbers of this form. We will prove that $a < q < b$. Now $q > a$ since $q - a = \frac{1}{\sqrt{p}}(b-a) > 0$ and $q < b$ since

$$b - q = (b-a)\left(1 - \frac{1}{\sqrt{p}}\right) > 0$$

and the result is established. □

The next result goes a little further than Theorem 1.2.1:

Theorem 1.2.3. *If N is a natural number then either it is a perfect square or $\sqrt{N}$ is irrational.*

Proof. Suppose that N is not a perfect square, and that $\sqrt{N}$ is a rational number. We seek a contradiction. Write $\sqrt{N}$ as a (non-negative) integer plus a fraction in its lowest terms i.e.

$$\sqrt{N} = a + \frac{b}{c}, \tag{1.2.2}$$

where $a, b, c \in \mathbb{N}, b < c$ and $\dfrac{b}{c}$ is expressed in its lowest terms. Multiply both sides of (1.2.2) by c, and then square both sides to get

$$\begin{aligned} c^2 N &= (ca+b)^2 \\ &= c^2a^2 + 2cab + b^2. \end{aligned}$$

Rearranging, we get

$$\begin{aligned} b^2 &= c^2N - c^2a^2 - 2cab \\ &= c(cN - ca^2 - 2ab). \end{aligned}$$

Then c is a factor of b^2, so that $b^2 = cd$ where $d = cN - ca^2 - 2ab \in \mathbb{N}$. So $c = \dfrac{b^2}{d}$. Substitute for c in (1.2.2) to get

$$\sqrt{N} = a + \frac{d}{b}. \tag{1.2.3}$$

We know $b < c$. We also have $d < b$, as $b^2 = dc > bd \Rightarrow b > d$. So $\dfrac{d}{b} = \dfrac{b}{c}$ is written in lower terms, and that is our desired contradiction. □

From this, we see that $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \ldots$ are irrational.

In MAS114, it was shown that $\mathbb{Q}$ is *countable*, (i.e. it can be out into one–to–one correspondence with $\mathbb{N}$), but that $\mathbb{R}$ is uncountable.[1] Since the union of two countable sets is countable, it follows that the set $\mathbb{Q}^c$ of irrationals is also uncountable.

There are many examples of real numbers for which it is still an open question as to whether they are rational, or irrational, e.g. $n_1 e + n_2 \pi$, for $n_1, n_2 \in \mathbb{Z} \setminus \{0\}$.

1.3 Inequalities

Inequalities are vital tools in analysis. You've already met some important examples of these in MAS111. Many of these involve the *modulus* $|a|$ of a real number a, which measures its size, or absolute value:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

The following alternative description can be quite useful:

$$|a| = \max\{a, -a\}.$$

We'll often use the (easily verified) fact that $|ab| = |a|.|b|$ for all $a, b \in \mathbb{R}$.

One of the most important inequalities you'll meet is the *triangle inequality*:

$$|a + b| \leq |a| + |b|,$$

for all $a, b \in \mathbb{R}$. You can remind yourself of how to prove this in Problem 4.

If $a, b \in \mathbb{R}$, the number $|a - b|$ is important, as it measures the distance between a and b, e.g. $|-7-1| = 8, |-7-(-1)| = 6$. But beware that "$|a - b| \leq |a| - |b|$" is NOT true (e.g. try $a = 1, b = 2$). The following result is sometimes called the "corollary to the triangle inequality":

Theorem 1.3.1. *For all $a, b \in \mathbb{R}$,*

$$||a| - |b|| \leq |a - b|.$$

[1] The proof of this uses the important *diagonal argument*, due to Georg Cantor (1845–1918).

Proof.

$$\begin{aligned}
|a| &= |a+0| \\
&= |a+(-b+b)| \\
&= |(a-b)+b| \\
&\leq |a-b|+|b|,
\end{aligned}$$

by the triangle inequality. Hence $|a|-|b| \leq |a-b|$. Now repeat the above argument, with the roles of a and b interchanged, to get

$$|b|-|a| \leq |b-a| = |a-b|.$$

Combining the two inequalities together, we have

$$||a|-|b|| = \max\{|a|-|b|, |b|-|a|\} \leq |a-b|,$$

as required. □

Another very famous inequality, which was proved in MAS111, is the *theorem of the means*: if $a_1, a_2, \ldots, a_n \geq 0$, then

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

In Problems 5 and 7, you can establish two other very useful inequalities:

- *Bernoulli's inequality*:

  $$(1+x)^n \geq 1 + nx,$$

  for all $n \in \mathbb{N}, x > -1$.

- *Cauchy's inequality*: if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers then

  $$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$

1.4 Bounds and Completeness

Let $A \subseteq \mathbb{R}$. We will use the convenient notation

$$-A = \{-a, a \in A\},$$

so e.g. if $A = \{-3, -2, 5\}$, then $-A = \{-5, 2, 3\}$.

Let $A \subset \mathbb{R}$ be finite and non-empty. We use $\max(A)$ and $\min(A)$ to denote the largest (smallest) numbers in A, respectively.[2] You might want to try to prove that $\min(A) = -\max(-A)$. We'll establish a more general result later on in this section.

We are also interested in infinite sets of real numbers. Let $a, b \in \mathbb{R}, a < b$. Important roles in analysis are played by the *closed intervals*:

$$[a, b] = \{x \in \mathbb{R}; a \leq x \leq b\},$$

and the *open intervals*:

$$(a, b) = \{x \in \mathbb{R}; a < x < b\}.$$

We also have infinite open intervals $(-\infty, a) = \{x \in \mathbb{R}; x < a\}$ and $(a, \infty) = \{x \in \mathbb{R}; x > a\}$. Half-open intervals such as $(a, b] = \{x \in \mathbb{R}; a < x \leq b\}$ and $[a, b) = \{x \in \mathbb{R}; a \leq x < b\}$ are also useful.

Observe that $[a, b]$ has a maximum element b, and a minimum element a.

Proposition 1.4.1. *The open interval* (a, b) *has no maximum or minimum element.*

Proof. Suppose that $x = \max((a, b))$. Then $x \in (a, b)$ so $x = b - \epsilon$ for some $\epsilon > 0$ (in fact, $0 < \epsilon < b - a$), But then $x < b - \epsilon/2 < b$, so $b - \epsilon/2 \in (a, b)$ is a larger number in the set than the maximum x, and we have a contradiction. The proof for the minimum is similar (Exercise). $\square$

We are now going to develop generalisations of maximum and minimum that are extremely useful tools in analysis. Let $A \subset \mathbb{R}$. We say that it is *bounded above* if there exists $M \in \mathbb{R}$ so that $x \leq M$ for all $x \in A$, *bounded below* if there exists $L \in \mathbb{R}$ such that $x \geq L$ for all $x \in A$, and *bounded* if it is both bounded above and below; e.g. $(-\infty, 1)$ is bounded above, but not below, $(-47, \pi)$ is bounded. If A is bounded above, the number M is called an *upper bound* for A, and if A is bounded below, then L is called a *lower bound* for A.

For $a, b \in \mathbb{R}$, the set (a, b) has many upper bounds, $b+1/2, b+1, b+2, b+3$ etc. but it is clear that b is special as it is the smallest. If A is an arbitrary non-empty subset of A that is bounded above, is it obvious that a least (i.e. smallest) upper bound for A exists? In fact we have the

Completeness property for $\mathbb{R}$. Every non-empty set of real numbers that is bounded above has a least upper bound.

[2] Formal definitions are $\max(A) := \{b \in A, b \geq a$ for all $a \in A\}$ and $\min(A) := \{c \in A, c \leq a$ for all $a \in A\}$.

For now, we will assume the completeness property. On the course website you can find a proof that the completeness property follows from the construction of the real numbers as limits of Cauchy sequences of rational numbers.[3]

If A is non-empty and bounded above, the least upper bound of A is usually called the *supremum*, and written $\sup(A)$. The alternative notation $\text{lub}(A)$ is sometimes used. We know that $\sup((a, b)) = b$ and if the set A has a maximum element, you can show (Exercise) that $\sup(A) = \max(A)$. It is worth emphasising that a real number $\alpha = \sup(A)$ if and only if

- α is an upper bound for the set A, i.e. $a \leq \alpha$ for all $a \in A$,
- If β is any other upper bound for A then $\beta > \alpha$.

The following property of the supremum can be rather useful. You should compare it to the definition of the limit of a sequence (see Chapter 2 and/or MAS111 notes). Remember that the Greek letter ϵ (epsilon) is frequently used to denote a real number that can be made "as small as you like".

Proposition 1.4.2. *If $A \subset \mathbb{R}$ is non-empty and bounded above, then given any $\epsilon > 0$, there exists $a \in A$ so that $a > \sup(A) - \epsilon$.*

Proof. Suppose that for some $\epsilon > 0$ such an a cannot be found, then for all $a \in A, a \leq \sup(A) - \epsilon$. But then $\sup(A) - \epsilon$ is a smaller upper bound for A than $\sup(A)$, and we have a contradiction. $\square$

Now suppose that A is non-empty and bounded below. In the next theorem, we will show that it has a greatest lower bound called the *infimum* and written $\inf(A)$ (or alternatively $\text{glb}(A)$). In particular $\inf((a, b)) = a$.

Theorem 1.4.3. *A set $A \subset \mathbb{R}$ is bounded below if and only if $-A$ is bounded above. Furthermore if $A \neq \emptyset$ is bounded below, then it has a greatest lower bound and*

$$\inf(A) = -\sup(-A).$$

Proof. It is easy to see that A is bounded below (by L say) if and only if $-A$ is bounded above (by $-L$). If $A \neq \emptyset$ is bounded below, then $-A$ is non-empty, and bounded above, so $\sup(-A)$ exists by the completeness property. Let $\alpha = -\sup(-A)$. Since for all $a \in A$, $-a \leq -\alpha$, then $a \geq \alpha$ and so α is a lower bound for A.

To show that α is the greatest lower bound, we'll use "proof by contradiction". So we assume that there exists $\beta \in \mathbb{R}$ so that $a \geq \beta > \alpha$ for all

[3] "Characterising Completeness of the Real Number Line", I suggest reading this after Chapter 2.

$a \in A$. Then $-a \leq -\beta < -\alpha$, i.e. $-\beta$ is a smaller upper bound for $-A$ than $-\alpha = \sup(-A)$, and that is our desired contradiction. $\square$

Once again, a real number $\gamma = \inf(A)$ if and only if

- γ is a lower bound for the set A, i.e. $a \geq \gamma$ for all $a \in A$,
- If δ is any other lower bound for A then $\delta < \gamma$.

The completeness property does not hold in $\mathbb{Q}$. For example, consider the set $A = \{q \in \mathbb{Q}; q^2 < 2\}$. It is non-empty, and bounded above by e.g. $3/2$, which is rational, but its supremum is $\sqrt{2}$, which is irrational. It has no supremum in the set $\mathbb{Q}$. We finish this chapter, by deducing some useful consequences of the completeness property. The next result is often called the *Archimedean property* of the real numbers, in honour of the great Archimedes of Syracuse (287 BCE – 212 BCE).

Theorem 1.4.4 (The Archimedean Property)**.** *Let x and y be arbitrary positive real numbers. Then there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof. Suppose that the statement is false so that $nx \leq y$ for all $n \in \mathbb{N}$. Then the set $A = \{nx; n \in \mathbb{N}\}$ is non-empty (as $x \in A$), and is bounded above by y. Hence, by the completeness property, A has a least upper bound. Write $\alpha = \sup(A)$ and choose any $n \in \mathbb{N}$. Then $(n+1)x \in A$ and so $(n+1)x \leq \alpha$. It follows that

$$nx \leq \alpha - x < \alpha.$$

As this argument works for any $n \in \mathbb{N}$, it follows that $\alpha - x$ is also an upper bound for A, which contradicts the fact that α is the smallest of these. $\square$

For applications of Theorem 1.4.4 to convergence of sequences, we often take $x = 1$ and y to be a large number. In fact, suppose that $\epsilon > 0$ is a very small real number. Then $1/\epsilon$ is very large, and Theorem 1.4.4 tells us that there exists $n \in \mathbb{N}$ so that $n > 1/\epsilon$. But then $1/n < \epsilon$. This last fact is very useful.

Some of the founders of calculus believed in the existence of "infinitesimals" – positive numbers dx that can be taken to be arbitrarily small. What does this mean? If dx exists, then $0 < dx < q$ for any positive rational number $q = m/n$, where $m, n \in \mathbb{N}$. But then $ndx < m$ for all $m, n \in \mathbb{N}$, and this contradicts the Archimedean property of $\mathbb{R}$, so we have shown that infinitesimals don't exist on the real number line.

We'll finish this section with a delightful and intriguing property of the real numbers. This is sometimes referred to as the *density* of the rational numbers in the real numbers.

Theorem 1.4.5. *Given any two real numbers x and y with $y > x$ there exists a rational number q such that*

$$x < q < y.$$

Proof. First assume that $x > 0$. As $y - x > 0$ we can apply Theorem 1.4.4 to find $n \in \mathbb{N}$ such that $n(y - x) > 1$, i.e.

$$nx + 1 < ny$$

Now apply Theorem 1.4.4 again to show that there exists $m \in \mathbb{N}$ such that $m > nx$. Hence the set

$$S = \{r \in \mathbb{N}; r > nx\} \neq \emptyset.$$

It follows by a theorem about natural numbers[4] that S has a minimum element p, and we must have $p - 1 \leq nx$. Then

$$ny > nx + 1 \geq p > nx,$$

and $q = p/n$ is the required rational number.

If $x = 0$, then $0 < y/2 < y$, and we may argue as before to find $y/2 < q < y$, which solves our problem.

If $x < 0$, then either $y > 0$, or $y \leq 0$. If $y > 0$, the same argument works as for the case $x = 0$.

If $y \leq 0$, then $0 \leq -y < -x$, and the rest is left to you. □

If we fix $x \in \mathbb{R}$, and take $y = x + \epsilon$, where $\epsilon > 0$ is very small, then Theorem 1.4.5 tells us that there exists $q \in \mathbb{Q}$ so that

$$|x - q| < \epsilon,$$

i.e. we can approximate an arbitrary real number as closely as we like by a rational number.

If x and y are both irrational numbers then Theorem 1.4.5 tells us that there exists a rational number q such that $x < q < y$, i.e. there is a rational number between every pair of irrationals. On the other hand, in Theorem 1.2.2, we showed that there are infinitely many irrational numbers between every pair of rationals. This seems to suggest that there are as many rationals as irrationals, but the set of all irrationals are uncountable, and so of a higher order of infinity than the set of all rationals. This gives us yet more insight into how complex, and counter–intuitive, the structure of the real number line is.

[4] This is sometimes called the *well-ordering principle.*

1.5 The Existence of $\sqrt{2}$

We happily use square roots. But how do we know they really exist? In this section, we'll prove that $\sqrt{2}$ exists in $\mathbb{R}$ as a consequence of the completeness property. To do this we'll need the following inequality: for $a, b > 0$ $(a \neq b)$:

$$\left(\frac{a+b}{2}\right)^2 > ab. \tag{1.5.4}$$

This is easily deduced by adding $4ab$ to both sides of $(a-b)^2 > 0$. It is also a special case of the theorem of the means (when $n = 2$), but since square roots appear explicitly there, it is better to derive it independently, as we have done.

Theorem 1.5.1. *There exists $\alpha \in \mathbb{R}$, with $\alpha > 0$, so that $\alpha^2 = 2$.*

Proof. Define $A = \{a \in \mathbb{R}; a^2 < 2\}$. $A \neq \emptyset$ as $1 \in A$, and A is bounded above, e.g. by 3, so by the completeness property $\alpha = \sup(A) \in \mathbb{R}$. As $\alpha > 1$, it is clearly positive. We must have either $\alpha^2 < 2, \alpha^2 > 2$ or $\alpha^2 = 2$. We'll show that the first two possibilities both yield contradictions, from which our result will follow. We will need to introduce $\beta = 2/\alpha$ and $x = (\alpha + \beta)/2$. From (1.5.4), for $\alpha \neq \beta$, we have $x^2 > \alpha\beta = 2$ and so

$$(2/x)^2 = 4/x^2 < 4/2 = 2,$$

i.e. $2/x \in A$.

First suppose $\alpha^2 < 2$. Then $\alpha < 2/\alpha$, i.e. $\alpha < \beta$. Hence $\alpha < x < \beta$ and so $2/x > 2/\beta = \alpha$. Since $2/x \in A$, this contradicts $\alpha = \sup(A)$.

Now suppose $\alpha^2 > 2$. This time (check) $\alpha > \beta$ and so $\beta < x < \alpha$. Since $x^2 > 2, x$ is an upper bound for A and this contradicts $\alpha = \sup(A)$.

$\square$