# Chapter 6

# Additional Material for MAS451/6352: Product Measures and Fubini's Theorem

These brief notes are intended as a short summary of additional reading that you are expected to do outside the lectures. They emphasise the main ideas and concepts, but you will need to work carefully through all the relevant proofs. The recommended source is Adams and Guillemin, section 2.5, pp.89-102. This material is examinable. It can be studied straight after Chapter 3.

The aim of this section is to learn more about product measures, and how the theory of Lebesgue integration deals with multiple (double, triple, etc.) integrals. Before delving further into the details of these ideas, we'll need an additional tool from the theory of $\sigma$-algebras.

## 6.1 Dynkin's $\pi - \lambda$ Lemma

Let $(S, \Sigma)$ be a measurable space. A collection $\mathcal{P}$ of sets in $S$ is called a $\pi$-*system* if $A, B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P}$ (i.e. $\mathcal{P}$ is closed under intersections).

A collection $\mathcal{L}$ of sets in $S$ is called a $\lambda$-*system* if

(L1) $S \in \mathcal{L}$.

(L2) If $(E_n)$ is an increasing sequence of sets in $\mathcal{L}$ (so $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$), then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{L}$.

(L3) If $E, F \in \mathcal{L}$ and $F \subset E$ then $E - F \in \mathcal{L}$.

Note that by (L1) and (L3), $\lambda$-systems are closed under complements.

**Proposition 6.1.1** *If $\mathcal{L}$ is a $\lambda$-system that is also a $\pi$-system, then it is a $\sigma$-algebra.*

*Proof.* Since $\mathcal{L}$ is closed under complements and finite intersections, it is closed under finite unions by de Morgan's laws. To show that $\mathcal{L}$ is a $\sigma$-algeba, we need to prove that if $(A_n)$ is an arbitrary sequence of sets in $\mathcal{L}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{L}$. This follows by writing $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$ and using (L2), where

$$B_1 = A_1, B_2 = B_1 \cup (A_2 - B_1), B_3 = B_2 \cup (A_3 - B_2) \ldots \square$$

Recall that if $\mathcal{A}$ is a collections of sets in $S$, then $\sigma(\mathcal{A})$ is the smallest $\sigma$-algebra which contains $\mathcal{A}$.

**Theorem 6.1.1 (Dynkin's $\pi - \lambda$ Lemma)** *If $\mathcal{P}$ is a $\pi$-system and $\mathcal{L}$ is a $\lambda$-system with $\mathcal{P} \subseteq \mathcal{L}$, then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.*

*Proof.* It suffices to prove that $\mathcal{L}(\mathcal{P})$, which is the smallest $\lambda$-system that contains $\mathcal{P}$, is a $\sigma$-algebra. By Proposition 6.1.1, its enough to prove that $\mathcal{L}(\mathcal{P})$ is closed under intersections.

*Step 1.* Fix $A \in \mathcal{L}(\mathcal{P})$ and define

$$\mathcal{G}_A = \{B \subseteq \Omega; A \cap B \in \mathcal{L}(\mathcal{P})\}.$$

You should check that $\mathcal{G}_A$ is a $\lambda$-system.

*Step 2.* If $A, B \in \mathcal{P}$, then $A \cap B \in \mathcal{P} \subseteq \mathcal{L}(\mathcal{P})$. Hence $B \in \mathcal{G}_A$. So we've shown that $\mathcal{P} \subseteq \mathcal{G}_A$, when $A \in \mathcal{P}$. But in Step 1, we proved that $\mathcal{G}_A$ is a $\lambda$-system, and so since $\mathcal{L}(\mathcal{P})$ is the smallest such, we deduce that $\mathcal{L}(\mathcal{P}) \subseteq \mathcal{G}_A$. This means in particular that if $A \in \mathcal{P}, B \in \mathcal{L}(\mathcal{P})$, then $A \cap B \in \mathcal{L}(\mathcal{P})$.

*Step 3.* If $A \in \mathcal{L}(\mathcal{P})$, then by Step 2 we have $A \cap B = B \cap A \in \mathcal{L}(\mathcal{P})$ when $B \in \mathcal{P}$. This shows that $\mathcal{P} \subseteq \mathcal{G}_A$, when $A \in \mathcal{L}(\mathcal{P})$. Now using Step 1 again, we find that $\mathcal{L}(\mathcal{P}) \subseteq \mathcal{G}_A$, and the result then follows. $\square$

This result can be very useful for proving that a function is measurable.

## 6.2 Product Measure

Let $(S_1, \Sigma_1, m_1)$ and $(S_2, \Sigma_2, m_2)$ be measure spaces.

The set $S_1 \times S_2 := \{(s_1, s_2); s_1 \in S_1, s_2 \in S_2\}$ is the usual Cartesian product of sets. If $A \subseteq S_1, B \subseteq S_2$, the subset $A \times B$ of $S_1 \times S_2$ is called a *product set*.

The $\sigma$-algebra $\Sigma_1 \otimes \Sigma_2$ is defined to be the smallest $\sigma$-algebra of subsets of $S_1 \times S_2$ which contains all the product sets.

If $E \subset S_1 \times S_2$ and $x \in S_1$, then $E_x \subset S_2$ is called an *$x$-slice* of $E$ where

$$E_x := \{y \in S_2; (x, y) \in E\}.$$

**Proposition 6.2.1** *If $E \in \Sigma_1 \otimes \Sigma_2$ then $E_x \in \Sigma_2$ for all $x \in S_1$.*

**Corollary 6.2.1** *Let $f : S_1 \times S_2 \to \mathbb{R}$ be measurable and fix $x \in S_1$. Define $f_x : S_2 \to \mathbb{R}$ by $f_x(y) = f(x, y)$ for all $y \in S_2$. Then $f_x$ is a measurable function.*

*Proof.* For all $a \in \mathbb{R}$, we need to show that $f_x^{-1}((a, \infty)) \in \Sigma_2$. Define $E = f^{-1}((a, \infty))$. Since $f$ is measurable, $E \in \Sigma_1 \otimes \Sigma_2$. By Proposition 6.2.1, $E_x \in \Sigma_2$. But $E_x = f_x^{-1}((a, \infty))$, and so the result follows. $\square$

We can similarly define $y$-slices of sets in $S_1 \times S_2$ and show that $f_y : S_1 \to \mathbb{R}$ is measurable, for any $y \in \mathbb{R}$, where $f_y(x) = f(x, y)$ for all $x \in S_1$.

We seek to define the product measure $m_1 \times m_2$ on $(S_1 \times S_2, \Sigma_1 \otimes \Sigma_2)$. We need to make an assumption about the measures that we are using. Let $(S, \Sigma, m)$ be a measure space. We say that the measure $m$ is $\sigma$-finite if there exists a sequence $(A_n)$ of subsets of $S$ with $A_n \in \Sigma$ for all $n \in \mathbb{N}$ such that $S = \bigcup_{n=1}^{\infty} A_n$ and $m(A_n) < \infty$ for all $n \in \mathbb{N}$. Clearly any finite measure (and hence all probability measures) are $\sigma$-finite. It is easy to see that Lebesgue measure on $\mathbb{R}$ is $\sigma$-finite. A measure space $(S, \Sigma, m)$ is said to be $\sigma$-finite, if $m$ is a $\sigma$-finite measure.

From now on, let $(S_1, \Sigma_1, m_1)$ and $(S_2, \Sigma_2, m_2)$ be $\sigma$-finite measure spaces. Let $E \in \Sigma_1 \otimes \Sigma_2$ and define $\phi_E : S_1 \to \mathbb{R}$ by

$$\phi_E(x) = m_2(E_x),$$

for all $x \in S_1$. By using the Dynkin $\pi - \lambda$ lemma, you can show that $\phi_E$ is measurable. Then we define product measure of $E$ by

$$(m_1 \times m_2)(E) = \int_{S_1} \phi_E(x) dm_1(x).$$

Again using the Dynkin $\pi - \lambda$ lemma, you can show that this definition is consistent, in that

$$(m_1 \times m_2)(E) = \int_{S_2} \psi_E(y) dm_2(y),$$

where $\psi_E : S_2 \to \mathbb{R}$ is the measurable function defined by $\psi_E(y) = m_1(E_y)$ for all $y \in S_2$.

In Problem 3 you can check that if $E = A \times B$ is a product set, then

$$(m_1 \times m_2)(A \times B) = m_1(A)m_2(B).$$

It can be shown that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ and Lebesgue measure on $\mathbb{R}^2$ is precisely $\lambda \times \lambda$.

## 6.3 Fubini's Theorem

We give two versions of this important result - one for general non-negative measurable functions and the other for integrable functions.

**Theorem 6.3.1 (Fubini's Theorem 1)** *Let $f : S_1 \times S_2 \to \mathbb{R}$ be a non-negative measurable function. Then the mappings*

$$x \to \int_{S_2} f(x,y)dm_2(y),$$

$$\textit{and} \;\; y \to \int_{S_1} f(x,y)dm_1(x),$$

*are both measurable. Furthermore*

$$\begin{aligned} \int_{S_1 \times S_2} fd(m_1 \times m_2) &= \int_{S_1} \left( \int_{S_2} f(x,y)dm_2(y) \right) dm_1(x) \\ &= \int_{S_2} \left( \int_{S_1} f(x,y)dm_1(x) \right) dm_2(y). \quad (6.3.1) \end{aligned}$$

The proof works by first establishing the result for indicator functions, and then extending by linearity to simple functions. The next step is to take an arbitrary non-negative measurable function, approximate it by simple functions as in Theorem 2.4.1 and use the monotone convergence theorem. Note that all three integrals in (6.3.1) may be infinite. The next result is more useful for applications.

**Theorem 6.3.2 (Fubini's Theorem 2)** *Let $f : S_1 \times S_2 \to \mathbb{R}$ be an integrable function. Then the mappings*

$$x \to \int_{S_2} f(x,y)dm_2(y),$$

$$\textit{and} \;\; y \rightarrow \int_{S_1} f(x,y) dm_1(x),$$

*are both equal (a.e.) to integrable functions. Furthermore*

$$\begin{aligned} \int_{S_1 \times S_2} f d(m_1 \times m_2) &= \int_{S_1} \left( \int_{S_2} f(x,y) dm_2(y) \right) dm_1(x) \\ &= \int_{S_2} \left( \int_{S_1} f(x,y) dm_1(x) \right) dm_2(y). \quad (6.3.2) \end{aligned}$$

The proof works by writing $f = f_+ - f_-$ and applying Theorem 6.3.1 to $f_-$ and $f_+$ separately.

All of the results of this chapter extend in a straightforward way to products of finitely many measure spaces.