Generalised Spherical Functions and Levy-Khinchine Formula on Groups and Symmetric Spaces

David Applebaum

School of Mathematics and Statistics, University of Sheffield, UK

Talk at *"Wales Mathematical Colloquium 2015"*, *Gregynog.*

18- 20 May 2015

Talk based on joint work with Tony Dooley (Bath)

Annales de l' Institut Henri Poincaré (Prob Stat) **51**, 599 (2015)

Outline of Talk

- Background on "structural" probability theory, the Lévy-Khintchine (LK) formula.
- Spherical functions on symmetric spaces.
- The LK formula for spherically symmetric measures.
- Generalised spherical functions and the LK formula for general measures.

Invitation to Structural Probability

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space. A *random variable* X is a measurable function from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$.

Invitation to Structural Probability

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space. A *random variable* X is a measurable function from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$. The *law* or *distribution* of X is the probability measure p_X on $(E, \mathcal{E})$, given by $p_X(A) = P(X^{-1}(A))$, for $A \in \mathcal{E}$.

Invitation to Structural Probability

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space.
A *random variable* X is a measurable function from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$.
The *law* or *distribution* of X is the probability measure p_X on $(E, \mathcal{E})$, given by $p_X(A) = P(X^{-1}(A))$, for $A \in \mathcal{E}$.
In elementary discrete probability we take $(E, \mathcal{E}) = (S, \mathcal{P}(S))$, where S is a finite set.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space.
A *random variable* X is a measurable function from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$.
The *law* or *distribution* of X is the probability measure p_X on $(E, \mathcal{E})$, given by $p_X(A) = P(X^{-1}(A))$, for $A \in \mathcal{E}$.
In elementary discrete probability we take $(E, \mathcal{E}) = (S, \mathcal{P}(S))$, where S is a finite set.
To study Brownian motion, Lévy processes etc. we take $(E, \mathcal{E}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Invitation to Structural Probability

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space.
A *random variable* X is a measurable function from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$.
The *law* or *distribution* of X is the probability measure p_X on $(E, \mathcal{E})$, given by $p_X(A) = P(X^{-1}(A))$, for $A \in \mathcal{E}$.
In elementary discrete probability we take $(E, \mathcal{E}) = (S, \mathcal{P}(S))$, where S is a finite set.
To study Brownian motion, Lévy processes etc. we take $(E, \mathcal{E}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
To investigate chance phenomena interacting with interesting mathematical structures, E might be a Banach space, a group or a manifold.

Let G be a locally compact Hausdorff group with identity e. Let $\mathcal{M}(G)$ be the set of all Borel (Radon) probability measures on G.

Let G be a locally compact Hausdorff group with identity e. Let $\mathcal{M}(G)$ be the set of all Borel (Radon) probability measures on G. The *convolution* of μ_1 and μ_2 in G is the unique $\mu_1 * \mu_2 \in \mathcal{M}(G)$ such that for all $f \in C_c(G)$:

$$\int_G f(g)(\mu_1 * \mu_2)(dg) = \int_G \int_G f(gh)\mu_1(dg)\mu_2(dh).$$

$(\mathcal{M}(G), *)$ is a monoid with unit δ_e (Dirac mass at e).

Let G be a locally compact Hausdorff group with identity e. Let $\mathcal{M}(G)$ be the set of all Borel (Radon) probability measures on G. The *convolution* of μ_1 and μ_2 in G is the unique $\mu_1 * \mu_2 \in \mathcal{M}(G)$ such that for all $f \in C_c(G)$:

$$\int_G f(g)(\mu_1 * \mu_2)(dg) = \int_G \int_G f(gh)\mu_1(dg)\mu_2(dh).$$

$(\mathcal{M}(G), *)$ is a monoid with unit δ_e (Dirac mass at e).

We say that $\mu \in \mathcal{M}(G)$ is *infinitely divisible* if for each $n \in \mathbb{N}$, there exists $\mu^{\frac{1}{n}} \in \mathcal{M}(G)$ so that

$$\mu = \mu^{\frac{1}{n}} * \mu^{\frac{1}{n}} * \cdots * \mu^{\frac{1}{n}}. \quad (n \text{ times})$$

Let G be a locally compact Hausdorff group with identity e. Let $\mathcal{M}(G)$ be the set of all Borel (Radon) probability measures on G. The *convolution* of μ_1 and μ_2 in G is the unique $\mu_1 * \mu_2 \in \mathcal{M}(G)$ such that for all $f \in C_c(G)$:

$$\int_G f(g)(\mu_1 * \mu_2)(dg) = \int_G \int_G f(gh)\mu_1(dg)\mu_2(dh).$$

$(\mathcal{M}(G), *)$ is a monoid with unit δ_e (Dirac mass at e).

We say that $\mu \in \mathcal{M}(G)$ is *infinitely divisible* if for each $n \in \mathbb{N}$, there exists $\mu^{\frac{1}{n}} \in \mathcal{M}(G)$ so that

$$\mu = \mu^{\frac{1}{n}} * \mu^{\frac{1}{n}} * \cdots * \mu^{\frac{1}{n}}. \quad (n \text{ times})$$

Let $G = \mathbb{R}^n$. *Fourier transform*: $\widehat{\mu}(y) = \int_{\mathbb{R}^n} e^{-ix\cdot y}\mu(dx)$, for all $y \in \mathbb{R}^n$.

The Lévy Khintchine Formula

Theorem

The measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ is infinitely divisible if and only if for all $y \in \mathbb{R}^n$,

$$\widehat{\mu}(y) = \exp\left(-\eta(y)\right),$$

The Lévy Khintchine Formula

Theorem

The measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ is infinitely divisible if and only if for all $y \in \mathbb{R}^n$,

$$\widehat{\mu}(y) = \exp\left(-\eta(y)\right),$$

where

$$\begin{aligned}\eta(y) &= ib \cdot y + \frac{1}{2}Ay \cdot y \\ &+ \int_{\mathbb{R}^n} (1 - e^{-iu \cdot y} - iu \cdot y \mathbf{1}_{B_1}(y))\nu(dy)\end{aligned}$$

where the *characteristics* (b, A, ν) are as follows:

where the *characteristics* (b, A, ν) are as follows:

- $b \in \mathbb{R}^n$,
- A is a non-negative definite, symmetric real-valued $n \times n$ matrix,
- The *Lévy measure* μ is a Borel (σ- finite) measure on $\mathbb{R}^n$ for which,

$$\nu(\{0\}) = 0 \quad , \quad \int_{\mathbb{R}^n} (1 \wedge |u|^2)\nu(du) < \infty.$$

where the *characteristics* (b, A, ν) are as follows:

- $b \in \mathbb{R}^n$,
- A is a non-negative definite, symmetric real-valued $n \times n$ matrix,
- The *Lévy measure* μ is a Borel (σ- finite) measure on $\mathbb{R}^n$ for which,

$$\nu(\{0\}) = 0 \quad , \quad \int_{\mathbb{R}^n} (1 \wedge |u|^2)\nu(du) < \infty.$$

Many important examples include Gaussian measure, Poisson measure, stable laws, t-distribution, χ^2-distribution, relativistic distribution, Riemann-zeta distribution etc. etc.

If μ is $O(n)$-invariant, i.e. for all $O \in O(n), A \in \mathcal{B}(\mathbb{R}^n)$,

$$\mu(OA) = \mu(A),$$

If μ is $O(n)$-invariant, i.e. for all $O \in O(n), A \in \mathcal{B}(\mathbb{R}^n)$,

$$\mu(OA) = \mu(A),$$

$$\eta(y) = -\frac{1}{2}a|u|^2 + \int_{\mathbb{R}^n}(1 - \cos(u \cdot y))\nu(dy),$$

If μ is $O(n)$-invariant, i.e. for all $O \in O(n), A \in \mathcal{B}(\mathbb{R}^n)$,

$$\mu(OA) = \mu(A),$$

$$\eta(y) = -\frac{1}{2}a|u|^2 + \int_{\mathbb{R}^n}(1 - \cos(u \cdot y))\nu(dy),$$

so μ has characteristics $(0, aI, \nu)$, where $a \geq 0$ and ν is an $O(n)$-invariant Lévy measure.

The LK formula extends, mutatis mutandis, after some work to

- *Locally Compact Hausdorff Abelian Groups G*,

  Here $\widehat{\mu}: \widehat{G} \to \mathbb{C}$ is given by $\widehat{\mu}(\chi) = \int_G \chi(g^{-1})\mu(dg)$,

The LK formula extends, mutatis mutandis, after some work to

- *Locally Compact Hausdorff Abelian Groups G*,

  Here $\widehat{\mu} : \widehat{G} \to \mathbb{C}$ is given by $\widehat{\mu}(\chi) = \int_G \chi(g^{-1})\mu(dg)$,
- *Real Banach spaces E*,

  Here $\widehat{\mu} : E^* \to \mathbb{C}$ is given by $\widehat{\mu}(y) = \int_E e^{-i\langle y,x\rangle}\mu(dx)$

The LK formula extends, mutatis mutandis, after some work to

- *Locally Compact Hausdorff Abelian Groups G*,

  Here $\widehat{\mu} : \widehat{G} \to \mathbb{C}$ is given by $\widehat{\mu}(\chi) = \int_G \chi(g^{-1})\mu(dg)$,
- *Real Banach spaces E*,

  Here $\widehat{\mu} : E^* \to \mathbb{C}$ is given by $\widehat{\mu}(y) = \int_E e^{-i\langle y,x\rangle}\mu(dx)$

Question: How can we extend this to non-abelian groups and to manifolds?

A (globally) *symmetric space* is a Riemannian manifold M such that each $p \in M$ is the fixed point of an involutive isometry s_p.

A (globally) *symmetric space* is a Riemannian manifold M such that each $p \in M$ is the fixed point of an involutive isometry s_p.
In fact, s_p is (locally) a geodesic symmetry.

Spherical Functions on Symmetric Spaces

A (globally) *symmetric space* is a Riemannian manifold M such that each $p \in M$ is the fixed point of an involutive isometry s_p.
In fact, s_p is (locally) a geodesic symmetry.
Every symmetric space M is a homogeneous space: $M = G/K$, where G is a Lie group, and K is a compact subgroup.

Spherical Functions on Symmetric Spaces

A (globally) *symmetric space* is a Riemannian manifold M such that each $p \in M$ is the fixed point of an involutive isometry s_p.
In fact, s_p is (locally) a geodesic symmetry.
Every symmetric space M is a homogeneous space: $M = G/K$, where G is a Lie group, and K is a compact subgroup.
In fact, G is the group of all isometries of M and K is the subgroup that leaves some point fixed.

Spherical Functions on Symmetric Spaces

A (globally) *symmetric space* is a Riemannian manifold M such that each $p \in M$ is the fixed point of an involutive isometry s_p.
In fact, s_p is (locally) a geodesic symmetry.
Every symmetric space M is a homogeneous space: $M = G/K$, where G is a Lie group, and K is a compact subgroup.
In fact, G is the group of all isometries of M and K is the subgroup that leaves some point fixed.
The canonical surjection $\pi : G \to G/K$ maps g to the left coset gK which is a point $p \in M$.

Examples include

- The n sphere $S^n = SO(n+1)/SO(n)$,
- Hyperbolic space $H^n = SO_0(n,1)/SO(n)$,
- Euclidean space $\mathbb{R}^n = M(n)/O(n)$, where the motion group $M(n) = \mathbb{R}^n \rtimes O(n)$,
- Real projective space $\mathbb{R}P(n) = SO(n+1)/O(n)$,
- Positive definite matrices $\mathcal{P}(n) = GL(n,\mathbb{R})/O(n)$.

Spherical Functions

Qn. How do we carry out harmonic analysis on M?

Spherical Functions

Qn. How do we carry out harmonic analysis on M?
A *spherical function* on G is a mapping $\Phi \in C^\infty(G, \mathbb{C})$ for which:

- $\Phi(e) = 1$,
- $\Phi(kgk') = \Phi(g)$ for all $g \in G, k, k' \in K$,
- Φ is an eigenfunction of all $D \in \mathcal{D}_K(G)$, the algebra of differential operators on G, that are left-invariant under G and right-invariant under K.

Spherical Functions

Qn. How do we carry out harmonic analysis on M?

A *spherical function* on G is a mapping $\Phi \in C^\infty(G, \mathbb{C})$ for which:

- $\Phi(e) = 1$,
- $\Phi(kgk') = \Phi(g)$ for all $g \in G, k, k' \in K$,
- Φ is an eigenfunction of all $D \in \mathcal{D}_K(G)$, the algebra of differential operators on G, that are left-invariant under G and right-invariant under K.

Equivalently Φ is spherical if it is a non-trivial complex-valued continuous function on G for which

$$\int_K \Phi(gkg')dk = \Phi(g)\Phi(g'),$$

for all $g, g' \in G$.

Equivalently Φ is spherical if it is a non-trivial complex-valued continuous function on G for which

$$\int_K \Phi(gkg')dk = \Phi(g)\Phi(g'),$$

for all $g, g' \in G$.
Here $dk = m(dk)$ is normalised Haar measure on K so that $m(K) = 1$ and for all $k, k' \in K, A \in \mathcal{B}(K)$,

$$m(kA) = m(Ak') = m(A).$$

We say that $\tilde{\Phi}$ is a *spherical function* on M if $\tilde{\Phi} \circ \pi$ is a spherical function on G.

Examples

- The two sphere S^2, $\Phi_m(\theta) = P_m(\cos(\theta))$ are *zonal spherical harmonics*, where $\theta \in [0, \pi)$ is the colatitude, and P_m is the Legendre polynomial of degree $m \in \mathbb{N}$.

We say that $\tilde{\Phi}$ is a *spherical function* on M if $\tilde{\Phi} \circ \pi$ is a spherical function on G.

Examples

- The two sphere S^2, $\Phi_m(\theta) = P_m(\cos(\theta))$ are *zonal spherical harmonics*, where $\theta \in [0, \pi)$ is the colatitude, and P_m is the Legendre polynomial of degree $m \in \mathbb{N}$.
- Hyperbolic space H^1, $\Phi_m(\theta) = P_m(\cosh(\theta))$, where θ is the geodesic distance of a point from the "origin" $K(0 \leq \theta < \infty)$.

In many examples, spherical functions can be expressed in terms of hypergeometric functions.

Spherical Measures, Spherical Transform

A Borel probability measure μ on G is *spherical* if for all $A \in \mathcal{B}(G), k, k' \in K$,

$$\mu(kAk') = \mu(A).$$

Spherical Measures, Spherical Transform

A Borel probability measure μ on G is *spherical* if for all $A \in \mathcal{B}(G), k, k' \in K$,

$$\mu(kAk') = \mu(A).$$

The space of all spherical probability measures on G is denoted $\mathcal{M}_K(G)$.

Spherical Measures, Spherical Transform

A Borel probability measure μ on G is *spherical* if for all $A \in \mathcal{B}(G), k, k' \in K$,

$$\mu(kAk') = \mu(A).$$

The space of all spherical probability measures on G is denoted $\mathcal{M}_K(G)$.
The *spherical transform* of $\mu \in \mathcal{M}_K(G)$ is

$$\widehat{\mu}(\Phi) = \int_G \Phi(g^{-1})\mu(dg),$$

where Φ is a spherical function.

Spherical Measures, Spherical Transform

A Borel probability measure μ on G is *spherical* if for all $A \in \mathcal{B}(G), k, k' \in K$,

$$\mu(kAk') = \mu(A).$$

The space of all spherical probability measures on G is denoted $\mathcal{M}_K(G)$.
The *spherical transform* of $\mu \in \mathcal{M}_K(G)$ is

$$\widehat{\mu}(\Phi) = \int_G \Phi(g^{-1})\mu(dg),$$

where Φ is a spherical function.
The space of all bounded spherical functions may be equipped with a locally compact topology, as it is the spectrum of the commutative Banach algebra (under convolution) $L^1(K \backslash G/K)$.

Spherical Measures, Spherical Transform

A Borel probability measure μ on G is *spherical* if for all $A \in \mathcal{B}(G), k, k' \in K$,

$$\mu(kAk') = \mu(A).$$

The space of all spherical probability measures on G is denoted $\mathcal{M}_K(G)$.
The *spherical transform* of $\mu \in \mathcal{M}_K(G)$ is

$$\widehat{\mu}(\Phi) = \int_G \Phi(g^{-1})\mu(dg),$$

where Φ is a spherical function.
The space of all bounded spherical functions may be equipped with a locally compact topology, as it is the spectrum of the commutative Banach algebra (under convolution) $L^1(K\backslash G/K)$.
For specific types of group G, there is a convenient parameterisation.

The Iwasawa Decomposition

From now on, let G be a *noncompact* Lie group (connected, semi-simple, finite centre) (e.g. $G = SL(2, \mathbb{R}), K = SO(2), M = H^1$).

We have the *Iwasawa decomposition*:

$$G \text{ is diffeomorphic to the direct product } KAN,$$

where A is abelian and N is nilpotent.

Let $\mathfrak{g} = T_e(G)$ be the *Lie algebra* of G. It is a linear, infinitesimal approximation to G and so contains important information about it.

Let $\mathfrak{g} = T_e(G)$ be the *Lie algebra* of G. It is a linear, infinitesimal approximation to G and so contains important information about it. The *exponential map* $\exp \mathfrak{g} \rightarrow G$. It is a diffeomorphism from a neighbourhood of 0 in $\mathfrak{g}$ to a neighbourhood of e in G.

Let $\mathfrak{g} = T_e(G)$ be the *Lie algebra* of G. It is a linear, infinitesimal approximation to G and so contains important information about it.
The *exponential map* $\exp \mathfrak{g} \rightarrow G$. It is a diffeomorphism from a neighbourhood of 0 in $\mathfrak{g}$ to a neighbourhood of e in G.
e.g. $G = SO(n), \mathfrak{g}$ is space of $n \times n$ skew-symmetric matrices, and exp is usual matrix exponential.

Let $\mathfrak{g} = T_e(G)$ be the *Lie algebra* of G. It is a linear, infinitesimal approximation to G and so contains important information about it.
The *exponential map* $\exp \mathfrak{g} \rightarrow G$. It is a diffeomorphism from a neighbourhood of 0 in $\mathfrak{g}$ to a neighbourhood of e in G.
e.g. $G = SO(n), \mathfrak{g}$ is space of $n \times n$ skew-symmetric matrices, and exp is usual matrix exponential.
At Lie algebra level, Iwasawa decomposition is:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

Let $\mathfrak{g} = T_e(G)$ be the *Lie algebra* of G. It is a linear, infinitesimal approximation to G and so contains important information about it.
The *exponential map* $\exp \mathfrak{g} \to G$. It is a diffeomorphism from a neighbourhood of 0 in $\mathfrak{g}$ to a neighbourhood of e in G.
e.g. $G = SO(n), \mathfrak{g}$ is space of $n \times n$ skew-symmetric matrices, and exp is usual matrix exponential.
At Lie algebra level, Iwasawa decomposition is:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

where $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ are Lie algebras of K, A, N (resp.) and at the global level, for $g \in G$,

$$g = u(g) \exp(A(g)) n(g),$$

where $u(g) \in K, A(g) \in \mathfrak{a}, n(g) \in \mathfrak{n}$.

The following is due to Harish-Chandra:

Theorem

Every spherical function on G is of the following form:

$$\Phi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))} dk,$$

where $g \in G$ and the parameter $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

Note that $\rho \in \mathfrak{a}^*$ is the famous half-sum of positive roots.

The following is due to Harish-Chandra:

Theorem

Every spherical function on G is of the following form:

$$\Phi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))} dk,$$

where $g \in G$ *and the parameter* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Note that $\rho \in \mathfrak{a}^*$ is the famous half-sum of positive roots.
In this context we can write the spherical transform:

$$\widehat{\mu}(\Phi_\lambda) = \widehat{\mu}(\lambda).$$

This is due to Gangolli (Acta. Math. (1964))

Theorem

$\mu \in \mathcal{M}_K(G)$ is infinitely divisible if and only if for all $\lambda \in \mathfrak{a}^*$,

$$\widehat{\mu}(\lambda) = \exp\left\{-\frac{1}{2}a\tau_\lambda - \int_G (1 - \Phi_\lambda(g))\nu(dg)\right\},$$

This is due to Gangolli (Acta. Math. (1964))

Theorem

$\mu \in \mathcal{M}_K(G)$ is infinitely divisible if and only if for all $\lambda \in \mathfrak{a}^*$,

$$\widehat{\mu}(\lambda) = \exp\left\{-\frac{1}{2}a\tau_\lambda - \int_G (1 - \Phi_\lambda(g))\nu(dg)\right\},$$

where ν is a bi-invariant Lévy measure, $a \geq 0$ and $-\tau_\lambda$ is the eigenvalue of the Laplace-Beltrami operator with eigenvector Φ_λ.

c.f. $G = \mathbb{R}, -\frac{d^2}{dx^2}e^{ixy} = y^2 e^{ixy}$.

This is due to Gangolli (Acta. Math. (1964))

Theorem

$\mu \in \mathcal{M}_K(G)$ is infinitely divisible if and only if for all $\lambda \in \mathfrak{a}^*$,

$$\widehat{\mu}(\lambda) = \exp\left\{-\frac{1}{2}a\tau_\lambda - \int_G (1 - \Phi_\lambda(g))\nu(dg)\right\},$$

where ν is a bi-invariant Lévy measure, $a \geq 0$ and $-\tau_\lambda$ is the eigenvalue of the Laplace-Beltrami operator with eigenvector Φ_λ.

c.f. $G = \mathbb{R}, -\frac{d^2}{dx^2}e^{ixy} = y^2 e^{ixy}$.

Problem: How to go beyond K-bi-invariance to arbitrary probability measures on G and G/K?

Beyond Spherical Functions I: Background Representation Theory

Let π be a *unitary representation* of a Lie group H in some complex Hilbert space V_π.

Beyond Spherical Functions I: Background Representation Theory

Let π be a *unitary representation* of a Lie group H in some complex Hilbert space V_π.
Then for all $g \in H, \pi(g)$ is a unitary operator on V_π such that

- $\pi(gh) = \pi(g)\pi(h)$, for all $g, h \in H$,
- The map $g \to \pi(g)\psi$ is continuous for all $\psi \in V_\pi$,
- $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$, for all $g \in H$,
- $\pi(e) = I$.

Beyond Spherical Functions I: Background Representation Theory

Let π be a *unitary representation* of a Lie group H in some complex Hilbert space V_π.
Then for all $g \in H, \pi(g)$ is a unitary operator on V_π such that

- $\pi(gh) = \pi(g)\pi(h)$, for all $g, h \in H$,
- The map $g \to \pi(g)\psi$ is continuous for all $\psi \in V_\pi$,
- $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$, for all $g \in H$,
- $\pi(e) = I$.

A closed subspace W_π of V_π is *invariant* for π if $\pi(g)W_\pi \subseteq W_\pi$ for all $g \in H$, and π is *irreducible* if its only invariant subspaces are V_π and $\{0\}$.

A closed subspace W_π of V_π is *invariant* for π if $\pi(g)W_\pi \subseteq W_\pi$ for all $g \in H$, and π is *irreducible* if its only invariant subspaces are V_π and $\{0\}$.

If π is irreducible so is the *conjugate representation* $\overline{\pi}$ on V_π^* defined by $\overline{\pi}(g) = J\pi(g)J^{-1}$ for all $g \in H$, where J is conjugate isomorphism between V_π and V_π^*.

A closed subspace W_π of V_π is *invariant* for π if $\pi(g)W_\pi \subseteq W_\pi$ for all $g \in H$, and π is *irreducible* if its only invariant subspaces are V_π and $\{0\}$.

If π is irreducible so is the *conjugate representation* $\overline{\pi}$ on V_π^* defined by $\overline{\pi}(g) = J\pi(g)J^{-1}$ for all $g \in H$, where J is conjugate isomorphism between V_π and V_π^*.

Every group H has a *trivial* (irreducible) representation π_0 acting on $\mathbb{C}$, by $\pi_0(g) = 1$ for all $g \in H$.

A closed subspace W_π of V_π is *invariant* for π if $\pi(g)W_\pi \subseteq W_\pi$ for all $g \in H$, and π is *irreducible* if its only invariant subspaces are V_π and $\{0\}$.

If π is irreducible so is the *conjugate representation* $\overline{\pi}$ on V_π^* defined by $\overline{\pi}(g) = J\pi(g)J^{-1}$ for all $g \in H$, where J is conjugate isomorphism between V_π and V_π^*.

Every group H has a *trivial* (irreducible) representation π_0 acting on $\mathbb{C}$, by $\pi_0(g) = 1$ for all $g \in H$.

The *unitary dual* $\widehat{H}$ of H is the set of all (equivalence classes modulo unitary equivalence) of irreducible representations.

Assume that H is compact, then

- For all irreducible representations π of H

$$d_\pi := \dim(V_\pi) < \infty,$$

so each $\pi(g)$ is a unitary matrix.

Assume that H is compact, then

- For all irreducible representations π of H

$$d_\pi := \dim(V_\pi) < \infty,$$

  so each $\pi(g)$ is a unitary matrix.
- $\widehat{H}$ is countable.

Assume that H is compact, then

- For all irreducible representations π of H

$$d_\pi := \dim(V_\pi) < \infty,$$

  so each $\pi(g)$ is a unitary matrix.
- $\widehat{H}$ is countable.

Theorem (Peter-Weyl)

$$L^2(H) = \bigoplus_{\pi \in \widehat{H}} \mathcal{M}_\pi,$$

where $\mathcal{M}_\pi :=$ *lin. span*$\{f^\pi_{u,v}; u, v \in V_\pi\}$ *and for all* $g \in H$*,*

$$f^\pi_{u,v}(g) := \sqrt{d_\pi}\langle \pi(g)u, v\rangle.$$

Define $\gamma_\pi : V_\pi \otimes V_\pi^* \to \mathcal{M}_\pi$ by

$$\gamma_\pi(u \otimes v^*) = f_{u,v}^\pi,$$

for all $u \in V_\pi, v^* \in V_\pi^*$.

Then γ_π extends to a unitary isomorphism.

Beyond Spherical Functions II: First Attempt at Generalisation

Return to non-compact G with compact subgroup K. Recall Harish-Chandra's formula for spherical functions:

$$\Phi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))} dk.$$

Beyond Spherical Functions II: First Attempt at Generalisation

Return to non-compact G with compact subgroup K. Recall Harish-Chandra's formula for spherical functions:

$$\Phi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))} dk.$$

In the harmonic analysis literature we meet *generalised spherical functions* or *Eisenstein integrals*:

$$\Phi_{\lambda,\pi}(g) = \int_K e^{(i\lambda+\rho)(A(kg))} \pi(k) dk,$$

where $\pi \in \widehat{K}$.

These are NOT the right tools for studying infinite divisibility.

Beyond Spherical Functions III: Second Attempt at Generalisation

What does work, are the *generalised Eisenstein integrals*:

$$\Phi_{\lambda,\pi_1,\pi_2}(g) = \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda+\rho)(A(kg))} \pi_1(u(kg)) \otimes \overline{\pi_2}(k) dk,$$

where $\pi_1, \pi_2 \in \widehat{K}$.

Beyond Spherical Functions III: Second Attempt at Generalisation

What does work, are the *generalised Eisenstein integrals*:

$$\Phi_{\lambda,\pi_1,\pi_2}(g) = \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda+\rho)(A(kg))} \pi_1(u(kg)) \otimes \overline{\pi_2}(k) dk,$$

where $\pi_1, \pi_2 \in \widehat{K}$.
It is helpful to consider this as an *infinite matrix* by ordering irreducibles $\pi_0, \pi_1, \pi_2, \ldots$ and writing

$$\Phi_\lambda(g) = (\Phi_{\lambda,\pi_i,\pi_j}(g)),$$

Beyond Spherical Functions III: Second Attempt at Generalisation

What does work, are the *generalised Eisenstein integrals*:

$$\Phi_{\lambda,\pi_1,\pi_2}(g) = \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda+\rho)(A(kg))} \pi_1(u(kg)) \otimes \overline{\pi_2}(k) dk,$$

where $\pi_1, \pi_2 \in \widehat{K}$.
It is helpful to consider this as an *infinite matrix* by ordering irreducibles $\pi_0, \pi_1, \pi_2, \ldots$ and writing

$$\Phi_\lambda(g) = (\Phi_{\lambda,\pi_i,\pi_j}(g)),$$

so the $(\pi_0, \pi_0)-$ entry is the Harish-Chandra spherical function, and the top row is the Eisenstein integral.

These are related to *principal series* of representations of G obtained using *Mackey's theory of induced representations*. So ξ_λ is a representation of G on $L^2(K)$ where for $g \in G, f \in L^2(K), l \in K$:

$$(\xi_\lambda(g)f)(l) = e^{(i\lambda+\rho)(A(lg))} f(u(lg)).$$

The representation is unitary if $\lambda \in \mathfrak{a}^*$.

These are related to *principal series* of representations of G obtained using *Mackey's theory of induced representations*. So ξ_λ is a representation of G on $L^2(K)$ where for $g \in G, f \in L^2(K), l \in K$:

$$(\xi_\lambda(g)f)(l) = e^{(i\lambda+\rho)(A(lg))}f(u(lg)).$$

The representation is unitary if $\lambda \in \mathfrak{a}^*$.
The connection is given by the following:

Theorem

For each $\lambda \in \mathfrak{a}^*, \pi_1, \pi_2 \in \widehat{K}, g \in G, u_1, v_1 \in V_{\pi_1}, u_2, v_2 \in V_{\pi_2}$,

$$\begin{aligned} & \langle \Phi_{\lambda,\pi_1,\pi_2}(g)(u_1 \otimes u_2^*), v_1 \otimes v_2^* \rangle_{V_{\pi_1} \otimes V_{\pi_2}^*} \\ = \; & \langle \xi_\lambda(g)\gamma_{\pi_1}(u_1 \otimes v_1^*), \gamma_{\pi_2}(u_2 \otimes v_2^*) \rangle_{L^2(K)}. \end{aligned}$$

Proof.

$$
\begin{aligned}
& \langle \xi_\lambda(g)\gamma_{\pi_1}(u_1 \otimes v_1^*), \gamma_{\pi_2}(u_2 \otimes v_2^*)\rangle_{L^2(K)} \\
= \; & \langle \xi_\lambda(g) f^{\pi_1}_{u_1,v_1}, f^{\pi_2}_{u_2,v_2}\rangle_{L^2(K)} \\
= \; & \int_K e^{(i\lambda+\rho)(A(kg))} f^{\pi_1}_{u_1,v_1}(u(kg))\overline{f^{\pi_2}_{u_2,v_2}(k)}dk \\
= \; & \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda+\rho)(A(kg))} \langle \pi_1(u(kg))u_1, v_1\rangle \overline{\langle \pi_2(k)u_2, v_2\rangle} dk \\
= \; & \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda+\rho)(A(kg))} \langle \pi_1(u(kg))u_1, v_1\rangle \langle \overline{\pi_2}(k)u_2^*, v_2^*\rangle dk \\
= \; & \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda+\rho)(A(kg))} \langle (\pi_1(u(kg)) \otimes \overline{\pi_2}(k))u_1 \otimes u_2^*, v_1 \otimes v_2^*\rangle dk \\
= \; & \langle \Phi_{\lambda,\pi_1,\pi_2}(g)(u_1 \otimes u_2^*), v_1 \otimes v_2^*\rangle_{V_{\pi_1} \otimes V^*_{\pi_2}} \qquad \square
\end{aligned}
$$

The next result demonstrates that the group composition rule manifests as matrix multiplication of generalised Eisenstein integrals:

Theorem

For all $\lambda \in \mathfrak{a}^*, \pi_1, \pi_2 \in \widehat{K}, g, h \in G, 1 \leq i, k \leq d_{\pi_1}, 1 \leq j, l \leq d_{\pi_2}$,

$$\Phi_{\lambda,\pi_1,\pi_2}^{(i,j),(k,l)}(gh) = \sum_{\eta \in \widehat{K}} \sum_{p,q=1}^{d_\eta} \Phi_{\lambda,\pi_1,\eta}^{(i,p),(k,q)}(h) \Phi_{\lambda,\eta,\pi_2}^{(p,j),(q,l)}(g) \qquad (1.1)$$

Proof.

$$\begin{aligned}
\Phi_{\lambda,\pi_1,\pi_2}^{(i,j),(k,l)}(gh) &= \sqrt{d_{\pi_1} d_{\pi_2}} \langle \xi(gh)(\langle \pi_1(\cdot)e_{\pi_1}^i, e_{\pi_1}^k \rangle), \langle \pi_2(\cdot)e_{\pi_2}^j, e_{\pi_2}^l \rangle \rangle_{L^2(K)} \\
&= \sqrt{d_{\pi_1} d_{\pi_2}} \langle \xi(h)(\langle \pi_1(\cdot)e_{\pi_1}^i, e_{\pi_1}^k \rangle), \xi(g^{-1})(\langle \pi_2(\cdot)e_{\pi_2}^j, e_{\pi_2}^l \rangle) \rangle_{L^2} \\
&= \sqrt{d_{\pi_1} d_{\pi_2}} \sum_{\eta \in \widehat{K}} d_\eta \sum_{p,q=1}^{d_\eta} \langle \xi(h)(\langle \pi_1(\cdot)e_{\pi_1}^i, e_{\pi_1}^k \rangle), \langle \eta(\cdot)e_\eta^p, e_\eta^q \rangle \rangle_{L} \\
&\times \langle \xi(g)(\langle \eta(\cdot)e_\eta^p, e_\eta^q \rangle), \langle \pi_2(\cdot)e_{\pi_2}^j, e_{\pi_2}^l \rangle \rangle_{L^2(K)}
\end{aligned}$$

and the result follows.

We can write (1.1) succinctly as

$$\Phi_{\lambda,\pi_1,\pi_2}(gh) = \sum_{\eta \in \widehat{K}} \Phi_{\lambda,\pi_1,\eta}(h)\Phi_{\lambda,\eta,\pi_2}(g) \tag{1.2}$$

The Eisenstein Transform

Let $\mu \in \mathcal{M}(G)$. Define its *Eisenstein transform* to be the matrix valued integral defined for each $\pi_1, \pi_2 \in \widehat{K}$ by:

$$\widehat{\mu}_{\lambda,\pi_1,\pi_2} := \int_G \Phi_{\lambda,\pi_1,\pi_2}(g^{-1})\mu(dg) \tag{1.3}$$

for all $\lambda \in \mathfrak{a}^*$.

The Eisenstein Transform

Let $\mu \in \mathcal{M}(G)$. Define its *Eisenstein transform* to be the matrix valued integral defined for each $\pi_1, \pi_2 \in \widehat{K}$ by:

$$\widehat{\mu}_{\lambda,\pi_1,\pi_2} := \int_G \Phi_{\lambda,\pi_1,\pi_2}(g^{-1})\mu(dg) \tag{1.3}$$

for all $\lambda \in \mathfrak{a}^*$.
This interacts perfectly with convolution of measures to give (in the sense of multiplication of infinite matrices):

$$(\widehat{\mu^{(1)} * \mu^{(2)}})_\lambda = \widehat{\mu_\lambda^{(1)}}\widehat{\mu_\lambda^{(2)}}.$$

The Lévy-Khintchine Formula

Theorem

If $\mu \in \mathcal{M}(G)$ is infinitely divisible (without idempotent factors) then (in the sense of infinite matrices) for all $\lambda \in \mathfrak{a}^*$,

$$\widehat{\mu}_\lambda = \mathit{Exp}(\psi_\lambda),$$

The Lévy-Khintchine Formula

Theorem

If $\mu \in \mathcal{M}(G)$ is infinitely divisible (without idempotent factors) then (in the sense of infinite matrices) for all $\lambda \in \mathfrak{a}^*$,

$$\widehat{\mu}_\lambda = Exp(\psi_\lambda),$$

$$\textit{where} \;\; \psi_\lambda = -b^i \rho_\lambda(X_i) + a^{ij} \rho_\lambda(X_i)\rho_\lambda(X_j) + \eta_\lambda.$$

The Lévy-Khintchine Formula

Theorem

If $\mu \in \mathcal{M}(G)$ is infinitely divisible (without idempotent factors) then (in the sense of infinite matrices) for all $\lambda \in \mathfrak{a}^*$,

$$\widehat{\mu}_\lambda = Exp(\psi_\lambda),$$

$$\textit{where} \;\; \psi_\lambda = -b^i \rho_\lambda(X_i) + a^{ij} \rho_\lambda(X_i)\rho_\lambda(X_j) + \eta_\lambda.$$

Here η_λ is the matrix whose diagonal entries are

$$\int_G (\Phi_{\lambda,\pi,\pi}(\tau^{-1}) - 1 + x^i(\tau)\rho_{\lambda,\pi,\pi}(X_i))\nu(d\tau)$$

and off-diagonal entries are

$$\int_G (\Phi_{\lambda,\pi_1,\pi_2}(\tau^{-1}) + x^i(\tau)\rho_{\lambda,\pi_1,\pi_2}(X_i))\nu(d\tau).$$

Here

- $\{X_1, X_2, \ldots, X_n\}$ is a basis for $\mathfrak{g}$,
- $b \in \mathbb{R}^n$, $a = (a_{ij})$ is a non-negative definite, symmetric $n \times n$ matrix, and ν is a Lévy measure on G,
- $x_1, \ldots, x_n$ are smooth functions on G which are canonical co-ordinates in a neighbourhood of e,
- For $X \in \mathfrak{g}$, $\rho_{\lambda,\pi_1,\pi_2}(X) := \frac{d}{dt}\Phi_{\lambda,\pi_1,\pi_2}(\exp(tX))\Big|_{t=0}$.

The proof works by first applying the *Dani-McCrudden embedding theorem* (see *Invent.Math.* **110** *237 (1992)*) to realise the infinitely divisible measure μ as ρ_1 in a convolution semigroup of probability measures $(\rho_t, t \geq 0)$.

The proof works by first applying the *Dani-McCrudden embedding theorem* (see *Invent.Math.* **110** *237 (1992)*) to realise the infinitely divisible measure μ as ρ_1 in a convolution semigroup of probability measures $(\rho_t, t \geq 0)$.

We then obtain a C_0 semigroup on $C_0(G)$ by the prescription:

$$T_t f(\sigma) = \int_G f(\sigma\tau)\rho_t(d\tau).$$

The proof works by first applying the *Dani-McCrudden embedding theorem* (see *Invent.Math.* **110** *237 (1992)*) to realise the infinitely divisible measure μ as ρ_1 in a convolution semigroup of probability measures $(\rho_t, t \geq 0)$.

We then obtain a C_0 semigroup on $C_0(G)$ by the prescription:

$$T_t f(\sigma) = \int_G f(\sigma\tau)\rho_t(d\tau).$$

The generators $\mathcal{L}$ of such semigroups were characterised by Hunt (1956). He showed that on a suitable domain of "twice-differentiable" functions

$$\mathcal{L}f(g) = b^i X_i f(g) + a^{ij} X_i X_j f(g) + \int_G (f(gh) - f(g) - x^i(h) X_i f(g))\nu(dh).$$

The proof works by first applying the *Dani-McCrudden embedding theorem* (see *Invent.Math.* **110** *237 (1992)*) to realise the infinitely divisible measure μ as ρ_1 in a convolution semigroup of probability measures $(\rho_t, t \geq 0)$.

We then obtain a C_0 semigroup on $C_0(G)$ by the prescription:

$$T_t f(\sigma) = \int_G f(\sigma\tau)\rho_t(d\tau).$$

The generators $\mathcal{L}$ of such semigroups were characterised by Hunt (1956). He showed that on a suitable domain of "twice-differentiable" functions

$$\mathcal{L}f(g) = b^i X_i f(g) + a^{ij} X_i X_j f(g) + \int_G (f(gh) - f(g) - x^i(h) X_i f(g))\nu(dh).$$

We then compute

$$\widehat{\mu}(\lambda) = \widehat{\rho_1}(\lambda) = T_1 \Phi_\lambda.$$

Diolch Yn Fawr Am Wrando.

Thank You For Listening.