Analytic and Probabilistic Perspectives on the Hardy-Littlewood-Sobolev Inequality

David Applebaum

School of Mathematics and Statistics, University of Sheffield, UK

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Outline of Talk

- Background on fractional integral operators. Work of Hardy and Littlewood.
- Analytic approach using ultracontractive semigroups. Simplified proof of Varopoulos' generalisation of Hardy-Littlewood-Sobolev.
- Probabilistic approach using Brownian motion and quadratic variation.

Historical Background on Fractional Integration

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This generalised to fractional order a formula due to Cauchy:

$$(I_n f)(x) = \int_0^x \int_0^{y_1} \cdots \int_0^{y_{n-1}} f(y_n) dy_n dy_{n-1} \cdots dy_1.$$

Riemann also introduced the *Riemann-Liouville fractional derivative*

$$D^{\alpha} = \frac{d}{dx} \circ I_{1-\alpha},$$

which enables the solution of Abel's integral equation, i.e. for given f, to find ϕ so that

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The case $\alpha = \frac{1}{2}$ is the "tautochrome" problem which finds the time taken for a particle to fall under gravity along a specified path f.

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Sobolev extended this to d - dimensions, where we have the *Riesz potential operator*, for $0 < \alpha < d$:

$$(I_\alpha f)(x) = \frac{1}{c_d} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy = (f * r_d)(x),$$

where the *Riesz kernel* is $r_d(x) = \frac{1}{c_d |x|^{d-\alpha}}$, and $c_d = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^{\frac{\alpha}{2}} \pi^{\frac{d}{2}}}$.

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Theorem (Hardy-Littlewood-Sobolev)

If $0 < \alpha < d, 1 < p < \frac{d}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, then there exists $C \geq 0$ so that for all $f \in L^p$,

$$||I_\alpha f||_q \leq C||f||_p. \quad ...(HLS)$$

Varopoulos' Theory

For $x \in \mathbb{R}^d$, $t \geq 0$, introduce *Gauss kernel*: $k(t;x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2t}}$, and the Gaussian semigroup

$$(T_t f)(x) = (f * k(t;\cdot))(x) = \int_{\mathbb{R}^d} f(y)k(t;x-y)dy,$$

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A straightforward calculation using the definition of $\Gamma(\alpha)$ shows that

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It follows that

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Let (S, Σ, μ) be a measure space and $(T_t, t \geq 0)$ be a C_0-contraction semigroup on $L^p(S)$ for $1 \leq p < \infty$. To simplify the proof, we assume that the semigroup is self-adjoint on $L^2(S)$. This holds in all interesting examples.

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We will need *Stein's maximal ergodicity theorem*:
For all $1 < p < \infty$ there exists $D_p > 0$ so that for all $f \in L^p(S)$,

$$||f^*||_p \leq D_p||f||_p, \qquad ...(S) \tag{1.1}$$

where for all $x \in S$, $f^*(x) = \sup_{t>0} |T_t f(x)|$.

The Ultracontractivity Assumption

We say that the semigroup $(T_t, t \geq 0)$ satisfies (n, p)*-ultracontractivity* if there exists an $n > 0$ (not required to be an integer) such that for all $1 \leq p < \infty$, there exists $C_{p,n} > 0$ so that for all $t > 0, f \in L^p(S)$,

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The number n will be referred to as the *dimension* of the semigroup. e.g. By Jensen's inequality, (n, p)-ultracontractivity holds whenever $T_t f(x) = \int_S f(y) k_t(x, y) \mu(dy)$ and k_t is a symmetric kernel satisfying $\int_S k_t(x, y) \mu(dy) = 1$, and an estimate of the form

$$k_t(x, y) \leq C t^{-\frac{n}{2}},$$

for all $t \geq 0, x, y \in S$.

Examples of Ultracontractivity

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For ultracontractive Schrödinger semigroups, see Davies and Simon, *J.Funct. Anal.* **59**, 335 (1984)

From now on, we define for $0 < \alpha < n$, and $(T_t, t \geq 0)$ an (n, p)-ultracontractive semigroup:

$$(I_\alpha f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1} T_t f(x) dt.$$

Its not hard to show that $(I_\alpha f)(x)$ is absolutely convergent.

We aim to prove (HLS). We proceed as follows:

Analytic Proof of HLS

Let $\delta > 0$ to be chosen later. Let $x \in S$ be arbitrary and choose $f \in L^1(S) \cap L^p(S)$ with $f \neq 0$.

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Using $f^*(x) := \sup_{t>0} |T_t f(x)|$ we have

$$|J_\alpha f(x)| \leq \frac{2}{\alpha} \frac{1}{\Gamma(\alpha/2)} f^*(x) \delta^{\frac{\alpha}{2}}.$$

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Now using (U) we obtain

$$\begin{aligned} |K_\alpha f(x)| &\leq C_{p,n,\alpha} \int_\delta^\infty t^{\frac{\alpha}{2} - \frac{n}{2p} - 1} ||f||_p \\ &\leq C_{p,n,\alpha} \delta^{\frac{\alpha}{2} - \frac{n}{2p}} ||f||_p, \end{aligned}$$

so that

$$|I_\alpha f(x)| \leq C_{p,n,\alpha}(f^*(x)\delta^{\frac{\alpha}{2}} + \delta^{\frac{\alpha}{2} - \frac{n}{2p}}||f||_p).$$

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Picking

$$\delta = \left(\frac{||f||_p}{f^*(x)}\right)^{2p/n}$$

to minimize the right hand side gives

$$|I_\alpha f(x)| \leq C_{p,n,\alpha}\left(f^*(x)\right)^{1-\alpha p/n}||f||_p^{\alpha p/n} = C_{p,n,\alpha}\left(f^*(x)\right)^{p/q}||f||_p^{\alpha p/n}.$$

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Thus for $1 < p < \frac{n}{\alpha}$ and using (S),

$$||I_\alpha f||_q^q \quad \leq \quad C_{p,n,\alpha}||f||_p^{\alpha pq/n}||f^*||_p^p$$

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and the required result follows by density.

Consequences of HLS 1: Fractional Powers

Let $-A$ be the (self-adjoint) infinitesimal generator of the semigroup $(T_t, t \geq 0)$ and assume that A is a positive operator in $L^2(S)$. For each $\gamma \in \mathbb{R}$, we can construct the self-adjoint operator A^γ in $L^2(S)$ by functional calculus, and we denote its domain in $L^2(S)$ by $\mathrm{Dom}(A^\gamma)$.

Theorem

For all $f \in Dom(A^{-\frac{\alpha}{2}})$,

$$I_\alpha(f) = A^{-\frac{\alpha}{2}} f,$$

in the sense of linear operators acting on $L^2(S)$

Proof.

We use the spectral theorem to write $T_t = \int_0^\infty e^{-t\lambda} P(d\lambda)$ for all $t \geq 0$ where $P(\cdot)$ is the projection-valued measure associated to A.

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$$
\begin{aligned}
\langle I_\alpha(f), g \rangle &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \int_0^\infty t^{\alpha/2-1} e^{-\lambda t} \langle P(d\lambda)f, g \rangle dt \\
&= \frac{1}{\Gamma(\alpha/2)} \left(\int_0^\infty t^{\alpha/2-1} e^{-t} dt \right) \left(\int_0^\infty \frac{1}{\lambda^{\frac{\alpha}{2}}} \langle P(d\lambda)f, g \rangle \right) \\
&= \langle A^{-\frac{\alpha}{2}} f, g \rangle.
\end{aligned}
$$

□

Consequences of HLS 2: Sobolev Inequality

Corollary

For all $1 < p < n, f \in Dom(A^{\frac{1}{2}})$, *if* $A^{\frac{1}{2}}f \in L^p(S)$ *then* $f \in L^{\frac{np}{n-p}}(S)$ *and*

$$||f||_{\frac{np}{n-p}} \leq C_{n,p,1}||A^{\frac{1}{2}}f||_p.$$

Proof.

Take $\alpha = 1$ so that so that $q = \frac{np}{n-p}$. Writing the Riesz potential operator as a fractional power in (HLS) yields $||A^{-\frac{1}{2}}f||_q \leq C_{n,p,1}||f||_p$.

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Take $\alpha = 1$ so that so that $q = \frac{np}{n-p}$. Writing the Riesz potential operator as a fractional power in (HLS) yields $||A^{-\frac{1}{2}} f||_q \leq C_{n,p,1} ||f||_p$. Replacing f with $A^{\frac{1}{2}} f$, we find that $||f||_q \leq C_{n,p,1} ||A^{\frac{1}{2}} f||_p$ as required. □

Remark.

1. In most cases of interest the operator A and the space S will be such that $\text{Dom}(A)^{\frac{1}{2}}$ contains a rich set of vectors such as Schwartz space (in $\mathbb{R}^d$) or the smooth functions of compact support (on a manifold). In practice, we would only apply the inequality to vectors in that set.
2. Note that in the case where $n > 2$ and $p = 2$ we have

$$||f||^2_{\frac{2n}{n-2}} \leq D\mathcal{E}(f),$$

where $\mathcal{E}(f) := \langle Af, f \rangle$ is a Dirichlet form. If S is a complete Riemannian manifold with bounded geometry and $-A$ is the Laplacian Δ, then we have $n = d$, the dimension of the manifold, and our Sobolev inequality is more familiar to those who are knowledgeable about that topic.

Probabilistic Approach

We return to the case $S = \mathbb{R}^d$. Then $T_t = e^{t\frac{\Delta}{2}}$ is the *heat semigroup*.

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Probabilistic Approach

We return to the case $S = \mathbb{R}^d$. Then $T_t = e^{t\frac{\Delta}{2}}$ is the *heat semigroup*. Our first goal is to find a probabilistic interpretation of the Riesz potential operator I_α:
Let $(B(t), t \geq 0)$ be standard Brownian motion on $\mathbb{R}^d$. It makes life easier if we use a non-standard notion for "expectation":

$$\begin{aligned} \mathbb{E}(f(B(t))) &:= \int_{\mathbb{R}^d} \mathbb{E}(f(B(t))|B(0) = x)dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)k(t, x - y)dydx \\ &= \int_{\mathbb{R}^d} f(y) \left(\int_{\mathbb{R}^d} k(t, x - y)dx \right) dy \\ &= \int_{\mathbb{R}^d} f(y)dy \qquad ...(E) \end{aligned}$$

which is intuitively the same as taking $B(0)$ to have Lebesgue measure as its "distribution."

Two Useful Martingales

For $f \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz space of rapidly decreasing functions) and fixed $a > 0$, consider the martingales:

$$M_f^a(t) = \int_0^{a \wedge t} \nabla(T_{a-s}f)(B_s) \cdot dB_s$$

$$M_f^{a,\alpha}(t) = \int_0^{a \wedge t} (a - s)^{\alpha/2} \nabla(T_{a-s}f)(B_s) \cdot dB_s.$$

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Quadratic variations:

$$[M_f^a](t) = \int_0^{a \wedge t} |\nabla(T_{a-s}f)(B_s)|^2 ds$$

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A Useful Identity

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$$f(B_a) = T_a f(B_0) + \int_0^a \nabla(T_{a-s}f)(B_s) \cdot dB_s.$$

If $g \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned} & g(B_a) \int_0^a (a-s)^{\alpha/2} \nabla(T_{a-s}f)(B_s) \cdot dB_s \\ = \; & T_a g(B_0) \int_0^a (a-s)^{\alpha/2} \nabla(T_{a-s}f)(B_s) \cdot dB_s \\ + \; & \left(\int_0^a \nabla(T_{a-s}g)(B_s) \cdot dB_s \right) \left(\int_0^a (a-s)^{\alpha/2} \nabla(T_{a-s}f)(B_s) \cdot dB_s \right). \end{aligned}$$

Observe further that the expectation of the first term is zero. That is,

$$\begin{aligned} & \mathbb{E}\left(T_a g(B_0)\int_0^a (a-s)^{\alpha/2}\nabla(T_{a-s}f)(B_s)\cdot dB_s\right) \\ = & \int_{\mathbb{R}^d}\mathbb{E}_x\left(T_a g(B_0)\int_0^a (a-s)^{\alpha/2}\nabla(T_{a-s}f)(B_s)\cdot dB_s\right)dx \\ = & \int_{\mathbb{R}^d} T_a g(x)\mathbb{E}_x\left(\int_0^a (a-s)^{\alpha/2}\nabla(T_{a-s}f)(B_s)\cdot dB_s\right)dx \\ = & \ 0 \end{aligned}$$

Thus by Itô's isometry,

$$\begin{aligned} & \mathbb{E}\left(g(B_a)\int_0^a (a-s)^{\alpha/2}\nabla(T_{a-s}f)(B_s)\cdot dB_s\right) \\ = & \ \mathbb{E}\left(\int_0^a \nabla(T_{a-s}g)(B_s)\cdot dB_s\right)\left(\int_0^a (a-s)^{\alpha/2}\nabla(T_{a-s}f)(B_s)\cdot dB_s\right) \\ = & \ \mathbb{E}\left(\int_0^a (a-s)^{\alpha/2}\nabla(T_{a-s}f)(B_s)\cdot\nabla(T_{a-s}g)(B_s)ds\right) \quad ...(ID) \end{aligned}$$

Probabilistic Interpretation of Riesz Potential Operator

From now on, let $t = a$ and define $M_f^{\alpha}(a) = M_f^{a,\alpha}(a)$.

Definition: Probabilistic Riesz Potential

Define for all $x \in \mathbb{R}^d$:

$$(\mathcal{S}^{a,\alpha} f)(x) = \mathbb{E}(M_f^{\alpha}(a) \mid B_a = x).$$

Theorem

For all $f \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d$,

1. $$\mathcal{S}^{a,\alpha} f(x) = -\int_0^a s^{\alpha/2} T_s(\Delta T_s f)(x) ds.$$

2. $$\lim_{a \to \infty} (\mathcal{S}^{a,\alpha} f)(x) = c_\alpha I_\alpha(f)(x),$$

   (almost everywhere) where $c_\alpha > 0$ depends only on α.

Proof. (1) Let $g \in \mathcal{S}(\mathbb{R}^d)$. Using (E) and (ID), we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \mathcal{S}^{a,\alpha} f(x) g(x) dx \\
= & \int_{\mathbb{R}^d} \mathbb{E}\left(\int_0^a (a-s)^{\alpha/2} \nabla(T_{a-s}f)(B_s) \cdot dB_s \mid B_a = x\right) g(x) dx \\
= & \ \mathbb{E}\left(\mathbb{E}\left(\int_0^a (a-s)^{\alpha/2} \nabla(T_{a-s}f)(B_s) \cdot dB_s \mid B_a\right) g(B_a)\right) \\
= & \ \mathbb{E}\left(\mathbb{E}\left(g(B_a) \int_0^a (a-s)^{\alpha/2} \nabla(T_{a-s}f)(B_s) \cdot dB_s\right) |B_a\right) \\
= & \ \mathbb{E}\left(g(B_a) \int_0^a (a-s)^{\alpha/2} \nabla(T_{a-s}f)(B_s) \cdot dB_s\right) \\
= & \ \mathbb{E}\left(\int_0^a (a-s)^{\alpha/2} \nabla(T_{a-s}f)(B_s) \cdot \nabla(T_{a-s}g)(B_s) ds\right) \\
= & \int_0^a \left\{ s^{\alpha/2} \int_{\mathbb{R}^d} \nabla(T_s f)(x) \cdot \nabla(T_s g)(x) dx \right\} ds
\end{aligned}$$

Now by integration by parts, and self-adjointness of the semigroup,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathcal{S}^{a,\alpha} f(x) g(x) dx \\
= &\ -\int_0^a \left\{ s^{\alpha/2} \int_{\mathbb{R}^d} T_s(\Delta(T_s f))(x) g(x) dx \right\} ds \\
= &\ -\int_{\mathbb{R}^d} \left\{ \int_0^a s^{\alpha/2}\, T_s(\Delta(T_s f))(x) ds \right\} g(x) dx.
\end{aligned}$$

(2) Recall that $\frac{d}{dt} T_t f = \Delta T_t f$. Write $u(t, \cdot) = T_t f$, then $\frac{\partial}{\partial t} u(t, \cdot) = \Delta u(t, \cdot)$ and so

$$\frac{\partial}{\partial t} u(2t, \cdot) = 2u'(2t, \cdot) = 2\Delta u(2t, \cdot).$$

This gives that

$$\Delta T_{2s} f = \frac{1}{2} \frac{d}{ds} T_{2s} f$$

Proof.

(2)
Hence

$$\begin{aligned}\mathcal{S}^{a,\alpha}f(x) &= -\int_0^a s^{\alpha/2}\Delta(T_{2s})f(x)ds \\ &= -\frac{1}{2}\int_0^a s^{\alpha/2}\frac{dT_{2s}f}{ds}(x)ds \\ &= -\frac{1}{2}a^{\alpha/2}T_{2a}f(x) + \frac{\alpha}{4}\int_0^a s^{\alpha/2-1}T_{2s}f(x)ds.\end{aligned}$$

Since by (U), $|T_{2a}f(x)| \leq \frac{C}{a^{d/2}}||f||_1$ and $0 < \alpha < d$, as $a \to \infty$, the right hand side of the previous equality goes to

$$\frac{\alpha}{4}\int_0^\infty s^{\alpha/2-1}T_{2s}f(x)ds = 2^{-\frac{\alpha+4}{2}}\alpha\Gamma(\alpha/2)I_\alpha f(x).$$

□

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Theorem

For all $x \in \mathbb{R}^d, t > 0$,

$$|\nabla_x k_t(x)| \leq 2^{\frac{d+4}{2}} \frac{1}{\sqrt{t}} k_{2t}(x). \quad ...(G)$$

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Proof. Observe that

$$\nabla_x k_t(x) = -\left(\frac{x_1}{t}, \cdots \frac{x_d}{t}\right) k_t(x)$$

so that

$$\begin{aligned} |\nabla_x k_t(x)| &\leq \frac{1}{\sqrt{t}}\sqrt{\frac{|x|^2}{t}} k_t(x) \\ &= \frac{1}{\sqrt{t}}\sqrt{\frac{|x|^2}{t}} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \end{aligned}$$

We now claim that the right hand side is dominated by $2^{\frac{d+4}{2}}\frac{1}{\sqrt{t}}k_{2t}(x)$.

To see this, observe that if $\sqrt{\frac{|x|^2}{t}} \leq 1$, then the right hand side is dominated by $\frac{1}{\sqrt{t}}\frac{1}{(2\pi t)^{d/2}}e^{-\frac{|x|^2}{2t}}$.

If $a = \sqrt{\frac{|x|^2}{t}} > 1$, then $a < a^2 = 4(a/2)^2 \leq 4e^{\frac{a^2}{4}}$ and the right hand side is dominated by

$$4\frac{1}{\sqrt{t}}\frac{1}{(2\pi t)^{d/2}}e^{(-\frac{|x|^2}{2t}+\frac{|x|^2}{4t})} = 4\frac{1}{\sqrt{t}}\frac{1}{(2\pi t)^{d/2}}e^{-\frac{|x|^2}{4t}}.$$

Since $e^{-\frac{|x|^2}{2t}} \leq e^{-\frac{|x|^2}{4t}}$, we see that in either case, the right hand side is dominated by

$$4\frac{1}{\sqrt{t}}\frac{1}{(2\pi t)^{d/2}}e^{-\frac{|x|^2}{4t}} = 2^{\frac{d+4}{2}}\frac{1}{\sqrt{t}}k_{2t}(x)$$

and this completes the proof.

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Recall that

$$[M_f^\alpha](a) = \int_0^{a \wedge t} (a - s)^\alpha |\nabla(T_{a-s}f)(B_s)|^2 ds.$$

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Lemma

Let $\delta > 0$ be arbitrary. Then there exists $C_1, C_2 \geq 0$ so that

$$[M_f^\alpha](a) \leq C_1 \left(\sup_{0<s<a} |(T_{2(a-s)}|f|)(B_s)| \right)^2 \delta^\alpha + C_2 ||f||_p^2 \delta^{\alpha - d/p} \quad \ldots(I)$$

The proof uses (G) and (U), and requires separate arguments for the two cases $\delta \geq a$ and $\delta < a$.

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If we minimise this inequality with respect to δ, just as in the case of the analytic proof, we find that there exists $C_{p,\alpha,d} > 0$ so that

$$[M_f^{\alpha}](a)^{\frac{1}{2}} \leq C_{p,\alpha,d} \left(\sup_{0<s<a} |(T_{2(a-s)}|f|)(B_s)| \right)^{p/q} ||f||_p^{\alpha p/d} \qquad ...(Q)$$

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Using martingale arguments, we can then deduce the following inequality: for $1 < p < \infty, f \in \mathcal{S}(\mathbb{R}^d)$ there exists $C_p > 0$ so that:

$$\left\| \left(\sup_{0<s<a} |(T_{2(a-s)}|f|)(B_s)| \right) \right\|_p \leq C_p \, ||f||_p ...(S)$$

Probabilistic Proof of HLS

Now we proceed to derive the classical HLS. We need the *Burkholder-Davis-Gundy inequality* for the martingale $(M_f^{a,\alpha}(t), t \geq 0)$ at $t = a$, namely there exists $C_q \geq 0$ so that

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Using the definition $(\mathcal{S}^{a,\alpha}f)(x) = \mathbb{E}(M_f^{\alpha}(a) \mid B_a = x)$, the fact that conditional expectation is an L^q-contraction and (Q) we deduce that:

$$\begin{aligned} ||\mathcal{S}^{a,\alpha}f||_q &\leq ||M_f^{\alpha}(a)||_q \\ &\leq C_q||[M_f^{\alpha}](a)^{\frac{1}{2}}||_q \\ &\leq C_{p,\alpha,d}\left\|\sup_{0<s<a}|(T_{2(a-s)}|f|)(B_s)|\right\|_p^{p/q}||f||_p^{\alpha p/d} \end{aligned}$$

Now apply (S) to find that

$$\begin{aligned} ||\mathcal{S}^{a,\alpha} f||_q &\leq C_{p,\alpha,d} ||f||_p^{\frac{p}{q}} ||f||_p^{\frac{\alpha p}{d}}. \\ &\leq C_{p,\alpha,d} ||f||_p, \end{aligned}$$

since

$$\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}.$$

Now apply (S) to find that

$$\begin{aligned} ||\mathcal{S}^{a,\alpha}f||_q &\leq C_{p,\alpha,d}||f||_p^{\frac{p}{q}}||f||_p^{\frac{\alpha p}{d}}. \\ &\leq C_{p,\alpha,d}||f||_p, \end{aligned}$$

since

$$\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}.$$

Since this bound does not depend on a, letting $a \to \infty$ and applying Fatou's lemma gives the result for $f \in \mathcal{S}(\mathbb{R}^d)$.
The result extends to all $f \in L^p$ by density of Schwartz space therein.

Thank You For Listening.

Diolch Yn Fawr Am Wrando.