Example 5.4 Revisited

We need to show that for $n > m$

$$d_1(f_n, f_m) = \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right)$$

where

$$f_n(x) = \begin{cases} 0 & \text{if } x \leqslant \frac{1}{2}, \\ n(x - \frac{1}{2}) & \text{if } \frac{1}{2} \leqslant x \leqslant \frac{1}{2} + \frac{1}{n}, \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} \leqslant x. \end{cases}$$

$$\begin{aligned} d_1(f_n, f_m) &= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \left[n\left(x - \frac{1}{2}\right) - m\left(x - \frac{1}{2}\right)\right] dx \\ &+ \int_{\frac{1}{2}+\frac{1}{n}}^{\frac{1}{2}+\frac{1}{m}} \left[1 - m\left(x - \frac{1}{2}\right)\right] dx \end{aligned}$$

Both integrals are straightforward to calculate. After some messy (but easy) algebra the first of these equals $\frac{1}{2n^2}(n - m)$ and the second equals $\frac{1}{2m} + \frac{m}{2n^2} - \frac{1}{n}$. Add these together and the result follows.