

A Generalised Gangolli-Lévy-Khintchine Formula for Infinitely Divisible Measures and Lévy Processes on Semi-Simple Lie Groups and Symmetric Spaces

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Abstract

In 1964 R. Gangolli published a Lévy-Khintchine type formula which characterised K bi-invariant infinitely divisible probability measures on a symmetric space G/K . His main tool was Harish-Chandra's spherical functions which he used to construct a generalisation of the Fourier transform of a measure. In this paper we use generalised spherical functions (or Eisenstein integrals) and extensions of these which we construct using representation theory to obtain such a characterisation for arbitrary infinitely divisible probability measures on a non-compact symmetric space. We consider the example of hyperbolic space in some detail.

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1 Introduction

A probability measure on a topological group G is said to be *infinitely divisible* if it has a convolution n th root for all natural numbers n . A stochastic process taking values in G is called a *Lévy process* if it has stationary and independent increments and is stochastically continuous. The probability laws of the process then form a (weakly continuous) convolution semigroup of measures. If G is a connected Lie group having at least one finite dimensional linear representation that has a discrete kernel then the *Dani-McCruden embedding theorem* states that every infinitely divisible measure is of the form μ_1 for some convolution semigroup of measures $(\mu_t, t \geq 0)$ (see [11, 12].)

The investigation of convolution semigroups (and hence Lévy processes) on Lie groups dates back to Hunt in 1956 who characterised them via the infinitesimal generators of the associated semigroup of operators acting on a Banach space of continuous functions on G [21]. Since then there has been much progress in developing understanding of these objects and the reader is directed to the monograph [24] for insight. In the case where $G = \mathbb{R}^d$, a more direct characterisation of infinitely divisible measures μ is obtained by using the Fourier transform to derive the classical *Lévy-Khintchine formula*

$$\int_{\mathbb{R}^d} e^{iu \cdot x} \mu(dx) = e^{-\eta(u)}, \quad (1.1)$$

where $\eta : \mathbb{R}^d \rightarrow \mathbb{C}$ is a continuous, hermitian negative-definite function (see e.g. [27, 4] for details.)

Since any globally Riemannian symmetric space M is a homogeneous space G/K where G is a connected semisimple Lie group and K is a compact subgroup, Lévy processes and infinitely divisible measures on M can be defined to be the images of those on G under the associated canonical surjection. Hence probability theory on M is reduced to that on the Riemannian symmetric pair (G, K) . In 1964 Gangolli [13] found a precise analogue of the Lévy-Khintchine formula (1.1) in this context for K bi-invariant infinitely divisible measures μ on G . A key ingredient was the generalisation of the Fourier transform in (1.1) to the spherical transform formed using a spherical function Φ on G (see e.g. [16] for background on these.) In fact we then have

$$\int_G \Phi(g) \mu(dg) = e^{-\eta_\Phi}, \quad (1.2)$$

where $\eta_\Phi \in \mathbb{C}$. If G is semisimple and M is irreducible then

$$\eta_\Phi = -a\lambda_\Phi + \int_{G-\{e\}} (1 - \Phi(g)) \nu(dg),$$

where λ_ϕ is an eigenvalue of the Laplace-Beltrami operator on G (corresponding to the eigenvector Φ), $a \geq 0$ and ν is a Lévy measure on $G - \{e\}$ (see also [2] and [25].)

Gangolli's paper [13] stimulated a great deal of further work on this subject. In [14] he investigated the sample paths of Lévy processes on M (see also [2].) Integrability and transience have been studied in [8], [9], [19] and [5]. Subordination on symmetric spaces was investigated in [3] and [1] while conditions for the existence of smooth densities were found in [25]. This work also inspired generalisations to general Riemannian manifolds [6], Gelfand pairs [18] and hypergroups [10].

We are particularly interested in the non-compact case. Then the spherical functions on G take the form

$$\Phi_\lambda(\sigma) = \int_K e^{(i\lambda+\rho)(A(k\sigma))} dk \quad (1.3)$$

for all $\sigma \in G$, where A is the abelian part of the Iwasawa decomposition of G , ρ is the half-sum of positive roots and the “parameter” λ runs through the complexification of the dual space of the abelian part of the Iwasawa decomposition of the Lie algebra of G (see below for more details.) Our goal in this paper is to extend Gangolli's result to general infinitely divisible measures on G (without any bi-invariance assumptions.) Spherical functions are no longer adequate tools for us to build the Fourier transform that we require. We note that these functions have been generalised to Eisenstein integrals (or generalised spherical functions) $\Phi_{\lambda,\pi}$ [17] wherein the measure dk in (1.3) is replaced by the “matrix-valued weight” $\pi(k)dk$ where π is an irreducible representation of the compact group K . In fact we find that this generalisation is not enough. To derive our formula we need to take a further step and consider a double parametrisation by the unitary dual \widehat{K} to form objects $(\Phi_{\lambda,\pi,\pi'}, \pi, \pi' \in \widehat{K})$ which we regard as infinite matrix valued functions defined on G . The construction of these *generalised Eisenstein integrals* utilises representation theory techniques and is carried out in section 3 of this paper. This part of the work may be of independent interest to specialists in harmonic analysis on Lie groups and symmetric spaces.

In section 5 of the paper we derive our Lévy-Khintchine type formula for semisimple groups. In fact we show that if μ is an infinitely divisible probability measure on G then

$$\int_G \Phi_\lambda(g)\mu(dg) = \text{Exp}(\psi_\lambda), \quad (1.4)$$

and we find a precise form for the infinite matrix $(\psi_{\lambda,\pi,\pi'}, \pi, \pi' \in \widehat{K})$. The meaning of the “exponential” Exp for an infinite matrix is made precise in section 4.

Our approach is different from Gangolli's. He worked directly with infinitely divisible measures. We work with convolution semigroups (in a similar way to [25] and [2]) so that we can take advantage of Hunt's classification [21] and first obtain (1.4) within that context. We then obtain the result for infinitely divisible measures from the Dani-McCrudden embedding theorem as described above. In section 6 we apply this theory to symmetric spaces G/K , and in section 7 we give a detailed description of the results on hyperbolic space, relating it to the Helgason Fourier transform. As pointed out to us by Peter Kim, these latter results are applicable to the statistical problem of deconvolution density estimation in hyperbolic space where they enable the assumption of spherical symmetry of the error density in [20] to be dropped. More generally, the results in this paper have potential for application to the study of transience and recurrence of Lévy processes in symmetric spaces (see e.g. [5]) and to the study of limiting properties and rates of convergence of components of Lévy processes in groups and symmetric spaces (see Chapters 6 and 7 of [24].)

As we are aiming this paper at an audience of both probabilists and harmonic analysts, we have included, in section 8 an appendix where we treat the properties of induced representations, principal series of semisimple groups and their K -types, which are needed as background for the present work. Whilst this material is in some senses known, it is hard to access in a uniform and consistent presentation. We hope that it will engender a wider readership of this work.

Notation. If G is a Lie group then $\mathcal{B}(G)$ is its Borel σ -algebra and $C_0(G)$ is the Banach space of real-valued continuous functions on G which vanish at infinity, equipped with the usual supremum norm. If T is a linear operator acting on $C_0(G)$ then $\text{Dom}(T)$ denotes its maximal domain. The $*$ -algebra of all bounded linear operators on a complex separable Hilbert space H is denoted $\mathcal{L}(H)$. Einstein summation convention is used throughout this paper. If $f : G \rightarrow \mathbb{C}$ is continuous we write $f^\#(g) = f(g^{-1})$ for $g \in G$. The (algebraic) dual of a complex vector space V will be denoted V^* .

2 Lévy Processes on Lie Groups

Let G be a Lie group with neutral element e , Lie algebra \mathfrak{g} and dimension n and let $Z = (Z(t), t \geq 0)$ be a G -valued stochastic process defined on some probability space (Ω, \mathcal{F}, P) . The right increment of Z between the times s and t where $s \leq t$ is the random variable $Z(s)^{-1}Z(t)$. We say that Z is a Lévy process on G if Z has stationary and independent right-increments, is stochastically continuous and $Z(0) = e$ (a.s.). Each $Z(t)$

has law μ_t and $(\mu_t, t \geq 0)$ is a weakly continuous convolution semigroup of probability measures with μ_0 being Dirac mass at e . Conversely given any such convolution semigroup we can always construct a Lévy process Z for which the law of $Z(t)$ is μ_t for each $t \geq 0$ on the space of all paths from $[0, \infty)$ to G by using Kolmogorov's construction theorem (see e.g. [5] and Theorem 10.5 on pp.55-7 of [27] for a detailed account of the case $G = \mathbb{R}^n$.) We refer the reader to [24] for a monograph treatment of Lie group valued Lévy processes. Given an arbitrary Lévy process Z we define an associated C_0 -contraction semigroup $(T_t, t \geq 0)$ on $C_0(G)$ by the prescription:

$$(T_t f)(g) = \mathbb{E}(f(gZ(t))) = \int_G f(gh)\mu_t(dh), \quad (2.5)$$

for each $f \in C_0(G), g \in G, t \geq 0$.

The infinitesimal generator \mathcal{L} of this semigroup was characterised by Hunt [21] in 1956 (see [24] pp.52-61 for a more recent treatment). We first fix a basis $(X_j, 1 \leq j \leq n)$ of \mathfrak{g} and define the dense linear manifold $C^2(G)$ by

$$C^2(G) = \{f \in C_0(G); X_i(f) \in C_0(G) \text{ and } X_i X_j(f) \in C_0(G) \text{ for all } 1 \leq i, j \leq n\},$$

where the Lie algebra acts as left-invariant vector fields.

There exist functions $x_i \in C_c^\infty(G), 1 \leq i \leq n$ so that (x_1, \dots, x_n) are a system of canonical co-ordinates for G at e . A measure ν defined on $\mathcal{B}(G - \{e\})$ is called a *Lévy measure* whenever

$$\int_{G-\{e\}} \left(\sum_{i=1}^n x_i(g)^2 \right) \nu(dg) < \infty \text{ and } \nu(U^c) < \infty,$$

for any neighbourhood U of e .

Theorem 2.1 (Hunt's theorem). *Let $(\mu_t, t \geq 0)$ be a weakly continuous convolution semigroup of measures in G with infinitesimal generator \mathcal{L} then*

1. $C^2(G) \subseteq \text{Dom}(\mathcal{L})$.
2. For each $g \in G, f \in C^2(G)$,

$$\mathcal{L}f(g) = b_i X_j f(g) + a^{ij} X_i X_j f(g) + \int_{G-\{e\}} (f(gh) - f(g) - x^i(h) X_i f(g)) \nu(dh), \quad (2.6)$$

where $b = (b^1, \dots, b^n) \in \mathbb{R}^n, a = (a^{ij})$ is a non-negative-definite, symmetric $n \times n$ real-valued matrix and ν is a Lévy measure on $G - \{e\}$.

Conversely, any linear operator with a representation as in (2.6) is the restriction to $C^2(G)$ of the infinitesimal generator of a semigroup of convolution operators on $C_0(G)$ that is induced by a unique weakly continuous convolution semigroup of probability measures.

From now on we assume that the Lie group G is connected, semi-simple and has a finite centre. Let K be a maximal compact subgroup of G with Lie algebra \mathfrak{k} and note that (G, K) is a non-compact Riemannian symmetric pair. The Iwasawa decomposition gives a global diffeomorphism between G and a direct product NAK where A and N are simply connected with A being abelian and N nilpotent wherein each $g \in G$ is mapped onto $n(g)\exp(A(g))u(g)$ where $u(g) \in K, n(g) \in N$ and $A(g) \in \mathfrak{a}$ which is the Lie algebra of A . Let \widehat{K} be the unitary dual of K , i.e. the set of all equivalence classes (modulo unitary transformations) of irreducible unitary representations of K . As is standard, we will frequently identify classes in \widehat{K} with representative elements. We recall that the *generalised spherical functions* or *Eisenstein integrals* [17] are the matrix-valued functions:

$$\Phi_{\lambda,\pi}(g) = \int_K e^{(i\lambda+\rho)(A(kg))} \overline{\pi}(k) dk \quad (2.7)$$

for $g \in G$, where $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ which is the complexification of the dual space \mathfrak{a}^* of \mathfrak{a} and ρ is half the sum of positive roots (relative to a fixed lexicographic ordering.) Note that $\overline{\pi}$ is the conjugate of $\pi \in \widehat{K}$ which we will discuss in detail within the next section.

One approach to generalising Gangolli's Lévy-Khintchine formula [13] (see also [2], [25]) would be to seek to find the (principal part of the) logarithm of

$$\alpha_{\lambda,\pi}(t) := T_t(\Phi_{\lambda,\pi})(e) = \int_G \Phi_{\lambda,\pi}(g) \mu_t(dg). \quad (2.8)$$

We will see that this plan is not adequate and that we need to extend (2.8) and consider it as a matrix valued function. We remark that some progress on the topic of general Lévy-Khintchine formulae on symmetric spaces was obtained by different methods in the PhD thesis of Han Zhang [34], however our results are more general and complete.

3 Representation Theoretic Aspects of Generalised Eisenstein Integrals

3.1 Generalised Eisenstein Integrals

In the appendix, we describe the construction of the principal series representations of a connected semisimple Lie group G . The reader can find more background to this section there. Define $M := Z_K(\mathfrak{a}) = \{k \in K; \text{Ad}(k)X = X \text{ for all } X \in \mathfrak{a}\}$. Then M is a compact Lie group whose Lie algebra is the centraliser of \mathfrak{a} in \mathfrak{k} . For a minimal parabolic subgroup NAM of G , we choose an irreducible unitary representation σ of M acting in a finite-dimensional vector space \mathcal{H}_σ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. The representation $1 \otimes \lambda \otimes \sigma$ of the minimal parabolic group NAM may be induced to a representation ξ_λ of $G = NAK$, which operates on $V_{K,\sigma} := \{f \in L^2(K, \mathcal{H}_\sigma) : f(mk) = \sigma(m)f(k), \text{ for all } m \in M \text{ and almost all } k \in K\}$. This representation is directly related (see (8.40) and preceding discussion for more details) to the following representation of G on $L^2(K)$:

$$(\xi'_\lambda(g)f)(l) = e^{(\lambda+\rho)(A(lg))} f(u(lg)), \quad (3.9)$$

for each $f \in V_{K,\sigma}, l \in K, g = n(g) \exp(A(g))u(g) \in G$.

We shall show in the appendix that for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, ξ'_λ may be considered as a representation of G in $L^2(K)$. If λ is pure imaginary then the representation is unitary (see (8.40) and the discussion that follows.) From now on we will take λ to be pure imaginary and write it as $i\lambda$ where $\lambda \in \mathfrak{a}^*$. Notice that the restriction of ξ' to K is precisely the right regular representation of K in $L^2(K)$. We will use this fact to decompose ξ' into its K -types.

For each $\pi \in \widehat{K}$, let V_π be the corresponding representation space so that for all $k \in K, \pi(k)$ acts unitarily in V_π . Note that V_π is finite-dimensional for each $\pi \in \widehat{K}$. For each $u, v \in V_\pi, k \in K$ define $f_{u,v}^\pi(k) := \sqrt{d_\pi} \langle \pi(k)u, v \rangle$ and let \mathcal{M}_π be the linear span of $\{f_{u,v}^\pi, u, v \in V_\pi\}$. Then \mathcal{M}_π is a closed finite-dimensional subspace of $L^2(K) := L^2(K, \mathbb{C})$ and by the Schur orthogonality relations, \mathcal{M}_π and $\mathcal{M}_{\pi'}$ are orthogonal if $\pi \neq \pi'$. By the Peter-Weyl theorem, $L^2(K) = \bigoplus_{\pi \in \widehat{K}} \mathcal{M}_\pi$.

Recall that the mapping $v \mapsto v^*$ is a conjugate linear bijection between V_π and V_π^* , where $v^*(u) = \langle u, v \rangle$ for $u, v \in V_\pi$, and that V_π^* becomes an inner product space when we define $\langle v^*, u^* \rangle_{V_\pi^*} = \langle u, v \rangle_{V_\pi}$. Then if $\pi \in \widehat{K}$, the *conjugate representation* is defined by

$$\langle \bar{\pi}(k)u^*, v^* \rangle_{V_\pi^*} = \langle \pi(k^{-1})v, u \rangle_{V_\pi} = \overline{\langle \pi(k)u, v \rangle_{V_\pi}}$$

for all $k \in K$.

For each $\pi \in \widehat{K}$, we define a linear mapping from $\gamma_\pi : V_\pi \otimes V_\pi^* \rightarrow \mathcal{M}_\pi$ by linear extension of the prescription

$$\gamma_\pi(u \otimes v^*) = f_{u,v}^\pi, \quad (3.10)$$

for each $u, v \in V_\pi$. The mapping γ_π is clearly well-defined.

Lemma 3.1. *For each $\pi \in \widehat{K}$, γ_π is a unitary isomorphism between $V_\pi \otimes V_\pi^*$ and \mathcal{M}_π . Moreover if $\pi_1, \pi_2 \in \widehat{K}$ with $\pi_1 \neq \pi_2$ then $\text{Ran}(\gamma_{\pi_2}) \perp \text{Ran}(\gamma_{\pi_1})$.*

Proof. It is clear from the definition (3.10) that γ_π is surjective and hence bijective. The rest follows by Schur orthogonality. Indeed for $u_1, v_1 \in V_{\pi_1}$, $u_2, v_2 \in V_{\pi_2}$,

$$\begin{aligned} \langle \gamma_{\pi_1}(u_1 \otimes v_1^*), \gamma_{\pi_2}(u_2 \otimes v_2^*) \rangle_{L^2(K)} &= \int_K f_{u_1, v_1}^{\pi_1}(k) \overline{f_{u_2, v_2}^{\pi_2}(k)} dk \\ &= \int_K \sqrt{d_{\pi_1} d_{\pi_2}} \langle \pi_1(k) u_1, v_1 \rangle_{V_{\pi_1}} \langle v_2, \pi_2(k) u_2 \rangle_{V_{\pi_2}} dk \\ &= \langle u_1, u_2 \rangle \langle v_2, v_1 \rangle \delta_{\pi_1, \pi_2} \\ &= \langle u_1 \otimes v_1^*, u_2 \otimes v_2^* \rangle \delta_{\pi_1, \pi_2}. \quad \square \end{aligned}$$

For each $\lambda \in \mathfrak{a}^*$, $\pi_1, \pi_2 \in \widehat{K}$, $g \in G$ define $\Phi_{\lambda, \pi_1, \pi_2}(g) \in \mathcal{L}(V_{\pi_1} \otimes V_{\pi_2}^*)$ by

$$\Phi_{\lambda, \pi_1, \pi_2}(g) := \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda + \rho)(A(kg))} (\pi_1(u(kg)) \otimes \overline{\pi_2(k)}) dk. \quad (3.11)$$

We call $\Phi_{\lambda, \pi_1, \pi_2}$ a *generalised Eisenstein integral*. If we take π_1 to be the trivial representation acting in \mathbb{C} then we may identify $\mathbb{C} \otimes V_{\pi_2}$ with V_{π_2} and then (3.11) yields $\Phi_{\lambda, \pi_2}(g) := \Phi_{\lambda, \pi_1, \pi_2}(g) = \sqrt{d_{\pi_2}} \int_K e^{(i\lambda + \rho)(A(kg))} \overline{\pi_2(k)} dk$ which is (a scalar multiple of) the Eisenstein integral (2.7). If π_2 is also taken to be trivial then (3.11) reduces to the usual spherical function on non-compact semisimple Lie groups. Eisenstein integrals were used by van den Ban [30] in his analysis of the principal series representations on reductive groups; more recently, van den Ban and Schlichtkrull [29] proved a full Plancherel decomposition for symmetric spaces using Eisenstein integrals. A lovely exposition of this work is given in [31] (see also [26]).

Now we come to a key structural result which gives the connection between group representations and generalised Eisenstein integrals (note that here, and in the sequel, we always write ξ instead of ξ' to keep the notation simple) :

Theorem 3.1. For each $\lambda \in \mathfrak{a}^*$, $\pi_1, \pi_2 \in \widehat{K}$, $g \in G$, $u_1, v_1 \in V_{\pi_1}$, $u_2, v_2 \in V_{\pi_2}$,

$$\langle \Phi_{\lambda, \pi_1, \pi_2}(g)(u_1 \otimes u_2^*), v_1 \otimes v_2^* \rangle_{V_{\pi_1} \otimes V_{\pi_2}^*} = \langle \xi_\lambda(g) \gamma_{\pi_1}(u_1 \otimes v_1^*), \gamma_{\pi_2}(u_2 \otimes v_2^*) \rangle_{L^2(K)} \quad (3.12)$$

Proof. By (3.10),

$$\begin{aligned} & \langle \xi_\lambda(g) \gamma_{\pi_1}(u_1 \otimes v_1^*), \gamma_{\pi_2}(u_2 \otimes v_2^*) \rangle_{L^2(K)} \\ &= \langle \xi_\lambda(g) f_{u_1, v_1}^{\pi_1}, f_{u_2, v_2}^{\pi_2} \rangle_{L^2(K)} \\ &= \int_K e^{(i\lambda + \rho)(A(kg))} f_{u_1, v_1}^{\pi_1}(u(kg)) \overline{f_{u_2, v_2}^{\pi_2}(k)} dk \\ &= \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda + \rho)(A(kg))} \langle \pi_1(u(kg)) u_1, v_1 \rangle \overline{\langle \pi_2(k) u_2, v_2 \rangle} dk \\ &= \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda + \rho)(A(kg))} \langle \pi_1(u(kg)) u_1, v_1 \rangle \langle \overline{\pi_2}(k) u_2^*, v_2^* \rangle dk \\ &= \sqrt{d_{\pi_1} d_{\pi_2}} \int_K e^{(i\lambda + \rho)(A(kg))} \langle (\pi_1(u(kg)) \otimes \overline{\pi_2}(k)) u_1 \otimes u_2^*, v_1 \otimes v_2^* \rangle dk \\ &= \langle \Phi_{\lambda, \pi_1, \pi_2}(g)(u_1 \otimes u_2^*), v_1 \otimes v_2^* \rangle_{V_{\pi_1} \otimes V_{\pi_2}^*} \quad \square \end{aligned}$$

We will want to work with matrix elements of $\Phi_{\lambda, \pi_1, \pi_2}(\cdot)$. To this effect let $(e_\pi^i, 1 \leq i \leq d_\pi)$ be an orthonormal basis in V_π . For ease of notation we will always use $(h_\pi^i, 1 \leq i \leq d_\pi)$ for the corresponding dual orthonormal basis in V_π^* so that $h_\pi^i := (e_\pi^i)^*$ for all $1 \leq i \leq d_\pi$. Define for all $\lambda \in \mathfrak{a}^*$, $\pi_1, \pi_2 \in \widehat{K}$, $g \in G$, $1 \leq i, k \leq d_{\pi_1}$, $1 \leq j, l \leq d_{\pi_2}$,

$$\Phi_{\lambda, \pi_1, \pi_2}^{(i, j), (k, l)}(g) := \langle \Phi_{\lambda, \pi_1, \pi_2}(g)(e_{\pi_1}^i \otimes h_{\pi_2}^j), e_{\pi_1}^k \otimes h_{\pi_2}^l \rangle, \quad (3.13)$$

where we emphasise that the indices i and k are associated with the representation π_1 while j and l are associated with π_2 . So we can identify the linear operator $\Phi_{\lambda, \pi_1, \pi_2}(g)$ acting on the finite dimensional vector space $V_{\pi_1} \otimes V_{\pi_2}^*$ with the $d_{\pi_1} d_{\pi_2} \times d_{\pi_1} d_{\pi_2}$ matrix $(\Phi_{\lambda, \pi_1, \pi_2}^{(i, j), (k, l)}(g))$ in the usual manner. It follows from (3.13) and (3.12) that the mapping $g \rightarrow \Phi_{\lambda, \pi_1, \pi_2}(g)$ from G to $V_{\pi_1} \otimes V_{\pi_2}^*$ is C^∞ . We equip $V_{\pi_1} \otimes V_{\pi_2}^*$ with the usual Euclidean norm.

Theorem 3.2. For all $\lambda \in \mathfrak{a}^*$, $\pi_1, \pi_2 \in \widehat{K}$, $g \in G$, $\frac{1}{d_{\pi_1} d_{\pi_2}} \Phi_{\lambda, \pi_1, \pi_2}(g)$ is a contraction in $V_{\pi_1} \otimes V_{\pi_2}^*$.

Proof. Using (3.12), the Cauchy-Schwarz inequality and Lemma 3.1 we find that for all $1 \leq i, k \leq d_{\pi_1}$, $1 \leq j, l \leq d_{\pi_2}$, $g \in G$,

$$\begin{aligned} |\Phi_{\lambda, \pi_1, \pi_2}^{(i, j), (k, l)}(g)| &= |\langle \xi_\lambda(g) \gamma_{\pi_1}(e_{\pi_1}^i \otimes h_{\pi_2}^j), \gamma_{\pi_2}(e_{\pi_1}^k \otimes h_{\pi_2}^l) \rangle| \\ &\leq \|\gamma_{\pi_1}(e_{\pi_1}^i \otimes h_{\pi_2}^j)\| \cdot \|\gamma_{\pi_2}(e_{\pi_1}^k \otimes h_{\pi_2}^l)\| \\ &= \|e_{\pi_1}^i \otimes h_{\pi_2}^j\| \cdot \|e_{\pi_1}^k \otimes h_{\pi_2}^l\| = 1. \end{aligned}$$

So we have $\max_{1 \leq i, k \leq d_{\pi_1}, 1 \leq j, l \leq d_{\pi_2}} |\Phi_{\lambda, \pi_1, \pi_2}^{(i, j), (k, l)}(g)| \leq 1$. The result then follows by a standard matrix estimate. \square

The next result demonstrates that the group composition rule manifests as matrix multiplication of generalised Eisenstein integrals:

Theorem 3.3. *For all $\lambda \in \mathfrak{a}^*$, $\pi_1, \pi_2 \in \widehat{K}$, $g, h \in G$, $1 \leq i, k \leq d_{\pi_1}$, $1 \leq j, l \leq d_{\pi_2}$,*

$$\Phi_{\lambda, \pi_1, \pi_2}^{(i, j), (k, l)}(gh) = \sum_{\eta \in \widehat{K}} \sum_{p, q=1}^{d_\eta} \Phi_{\lambda, \pi_1, \eta}^{(i, p), (k, q)}(h) \Phi_{\lambda, \eta, \pi_2}^{(p, j), (q, l)}(g) \quad (3.14)$$

Proof. By (3.13) and Theorem 3.1

$$\begin{aligned} \Phi_{\lambda, \pi_1, \pi_2}^{(i, j), (k, l)}(gh) &= \sqrt{d_{\pi_1} d_{\pi_2}} \langle \xi(gh) (\langle \pi_1(\cdot) e_{\pi_1}^i, e_{\pi_1}^k \rangle), \langle \pi_2(\cdot) e_{\pi_2}^j, e_{\pi_2}^l \rangle \rangle_{L^2(K)} \\ &= \sqrt{d_{\pi_1} d_{\pi_2}} \langle \xi(h) (\langle \pi_1(\cdot) e_{\pi_1}^i, e_{\pi_1}^k \rangle), \xi(g^{-1}) (\langle \pi_2(\cdot) e_{\pi_2}^j, e_{\pi_2}^l \rangle) \rangle_{L^2(K)} \\ &= \sqrt{d_{\pi_1} d_{\pi_2}} \sum_{\eta \in \widehat{K}} d_\eta \sum_{p, q=1}^{d_\eta} \langle \xi(h) (\langle \pi_1(\cdot) e_{\pi_1}^i, e_{\pi_1}^k \rangle), \langle \eta(\cdot) e_\eta^p, e_\eta^q \rangle \rangle_{L^2(K)} \\ &\quad \times \langle \xi(g) (\langle \eta(\cdot) e_\eta^p, e_\eta^q \rangle), \langle \pi_2(\cdot) e_{\pi_2}^j, e_{\pi_2}^l \rangle \rangle_{L^2(K)} \end{aligned}$$

and the result follows, again by Theorem 3.1. \square

We can write (3.14) succinctly as

$$\Phi_{\lambda, \pi_1, \pi_2}(gh) = \sum_{\eta \in \widehat{K}} \Phi_{\lambda, \pi_1, \eta}(h) \Phi_{\lambda, \eta, \pi_2}(g) \quad (3.15)$$

Before we continue with our study of generalised Eisenstein integrals, we first recall that if S is a countable set and $(H_s, s \in S)$ is a family of complex separable Hilbert spaces then we may form the direct sum $H := \bigoplus_{s \in S} H_s$ comprising vectors $x = \bigoplus_{s \in S} x_s$ (where each $x_s \in H_s$) for which $\sum_{s \in S} \|x_s\|^2 < \infty$. If $(T_s, s \in S)$ is a collection of linear operators with $T_s \in \mathcal{L}(H_s)$ for all $s \in S$ and $\sup_{s \in S} \|T_s\| < \infty$ then it is easy to verify that $T := \bigoplus_{s \in S} T_s \in \mathcal{L}(H)$ where

$$Tx := \bigoplus_{s \in S} T_s x_s.$$

Note that if T_s is a contraction for all $s \in S$ then so is T . More generally if each T_s is a densely defined linear operator with domain D_s then we may still define the direct sum operator T as above on the dense domain given by the linear subspace D of the vector space direct sum $\bigoplus_{s \in S} D_s$ comprising

sequences of vectors that vanish for all but finitely many entries. In particular if T and V are two such operators and V leaves D invariant, then TV is another such operator and

$$TV = \bigoplus_{s \in S} T_s V_s. \quad (3.16)$$

Let $\mathcal{H}(K) := \bigoplus_{\pi_1, \pi_2 \in \widehat{K}} V_{\pi_1} \otimes V_{\pi_2}^*$ and define for all $\lambda \in \mathfrak{a}^*, g \in G$ a linear operator $\Phi_\lambda(g)$ acting on the dense domain $V(K)$ of $\mathcal{H}(K)$ comprising sequences in $\bigoplus_{\pi_1, \pi_2 \in \widehat{K}} V_{\pi_1} \otimes V_{\pi_2}^*$ which vanish for all but finitely many entries by

$$\Phi_\lambda(g) = \bigoplus_{\pi_1, \pi_2 \in \widehat{K}} \Phi_{\lambda, \pi_1, \pi_2}(g).$$

Note that for all $g \in G$, $\Phi_\lambda(g)$ leaves $V(K)$ invariant.

Using (3.16), We can now interpret (3.15) as the composition of two linear operators:

$$\Phi_\lambda(gh) = \Phi_\lambda(h)\Phi_\lambda(g). \quad (3.17)$$

3.2 Infinitesimal Structure

We will also need some information about the differentials of the representations ξ_λ . To that end let θ be a Cartan involution on \mathfrak{g} and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let Σ_+ be the set of positive (with respect to a given lexicographic ordering) restricted roots. Let g_λ be the root space associated to the restricted root λ and define $X_\lambda \in g_\lambda$ by $\text{ad}(H)(X_\lambda) = \lambda(H)X_\lambda$ for all $H \in \mathfrak{a}$. It follows from the Iwasawa decomposition for semisimple Lie algebras (see e.g. [23], Proposition 6.4.3, p.373) that we can decompose each $X \in \mathfrak{p}$ as $X = H + Y_-$ where $H \in \mathfrak{a}$ and $Y_- = \sum_{\lambda \in \Sigma_+} (X_\lambda - \theta X_{-\lambda})$. Note that $Y_- \in \mathfrak{n}$ which is the Lie algebra of N . We will also find it useful to introduce $Y_+ = \sum_{\lambda \in \Sigma_+} (X_\lambda + \theta X_{-\lambda})$

Lemma 3.2. *Let $\lambda \in \mathfrak{a}^*$. If $X \in \mathfrak{k}$, then*

$$d\xi_\lambda(X) = X \quad (3.18)$$

and if $X \in \mathfrak{p}$, then

$$(d\xi_\lambda(X)f)(l) = \langle i\lambda + \rho, \Pi_{\mathfrak{a}}(\text{Ad}(l)X) \rangle f(l) + Y_+ f(l), \quad (3.19)$$

for $f \in C^\infty(K), l \in K$, where $\Pi_{\mathfrak{a}}$ is the orthogonal projection from \mathfrak{p} to \mathfrak{a} .

Proof. By (3.9) we have

$$(\xi_\lambda(\exp(tX)f))(l) = e^{(i\lambda+\rho)(A(l\exp(tX)))} f(u(lk_t)), \quad (3.20)$$

for each $t \in \mathbb{R}$, $f \in L^2(K)$, $l \in K$ and where we have used the Iwasawa decomposition $\exp(tX) = n_t a_t u(k_t)$.

We first suppose that $X \in \mathfrak{k}$ and so $\exp(tX) \in K$ for all $t \in \mathbb{R}$. Hence $A(l\exp(tX)) = 0$ for all $l \in K$. Consequently (3.20) takes the form

$$(\xi_\lambda(\exp(tX)f))(l) = f(l\exp(tX)),$$

and so (3.18) follows when take $f \in C^1(K)$, differentiate with respect to t and then put $t = 0$. Now suppose that $X \in \mathfrak{p}$. For each $H \in \mathfrak{a}$ we define $Y_{++}^{(H)} = \sum_{\lambda \in \Sigma_+} \lambda(H)(X_\lambda + \theta X_{-\lambda})$ and $Y_{--}^{(H)} = \sum_{\lambda \in \Sigma_+} \lambda(H)(X_\lambda - \theta X_{-\lambda})$. Note that $Y_+, Y_{++}^{(H)} \in \mathfrak{k}$ and $Y_{--}^{(H)} \in \mathfrak{n}$. Since each $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ it is easy to check that $[H, Y_-] = Y_{++}^{(H)}$ and $[H, Y_+] = Y_{--}^{(H)}$ for each $H \in \mathfrak{a}$. It follows by the Campbell-Baker-Hausdorff formula that for all $t \in \mathbb{R}$,

$$\exp(tX) = \exp(tH) \exp(tY_-) \exp\left(-\frac{1}{2}t^2 Y_{++}^{(H)}\right) \cdots,$$

from which we verify that $\left. \frac{d}{dt} A(\exp(tX)) \right|_{t=0} = H$.

It follows that

$$\left. \frac{d}{dt} \xi_\lambda(\exp(tX))f(l) \right|_{t=0} = \langle (i\lambda + \rho)\Pi_{\mathfrak{a}}(\text{Ad}(l)X) \rangle f(l). \quad (3.21)$$

We also have (by similar arguments) for $f \in C^\infty(K)$,

$$\left. \frac{d}{dt} f(u(lk_t)) \right|_{t=0} = Y_+ f(l). \quad (3.22)$$

(3.19) then follows from (3.21) and (3.22). \square

For each $\lambda \in \mathfrak{a}^*$, $\pi_1, \pi_2 \in \widehat{K}$, $X \in \mathfrak{g}$, $1 \leq i, k \leq d_{\pi_1}$, $1 \leq j, l \leq d_{\pi_2}$, we define

$$\rho_{\lambda, \pi_1, \pi_2}^{(i,j),(k,l)}(X) := \left. \frac{d}{dt} \Phi_{\lambda, \pi_1, \pi_2}^{(i,j),(k,l)}(\exp(tX)) \right|_{t=0} \quad (3.23)$$

Indeed it follows from (3.12) that $\rho_{\lambda, \pi_1, \pi_2}^{(i,j),(k,l)}(X)$ is well-defined and that

$$\rho_{\lambda, \pi_1, \pi_2}^{(i,j),(k,l)}(X) = \langle d\xi_\lambda(X) \gamma_{\pi_1}(e_{\pi_1}^i \otimes h_{\pi_1}^k), \gamma_{\pi_2}(e_{\pi_2}^j \otimes h_{\pi_2}^l) \rangle. \quad (3.24)$$

We will require the matrix $\rho_{\lambda, \pi_1, \pi_2} := (\rho_{\lambda, \pi_1, \pi_2}^{(i,j),(k,l)}(X))$ acting on $V_{\pi_1} \otimes V_{\pi_2}^*$ and also the direct sum operator $\rho_\lambda(X) := \bigoplus_{\pi_1, \pi_2 \in \widehat{K}} \rho_{\lambda, \pi_1, \pi_2}(X)$ acting on the dense domain $V(K)$. Note that operators of the form $\rho_\lambda(X)\Phi_\lambda(g)$ and $\rho_\lambda(X)\rho_\lambda(Y)\Phi_\lambda(g)$ are well-defined on the domain $V(K)$ for all $\lambda \in \mathfrak{a}^*$, $g \in G$, $X, Y \in \mathfrak{g}$ and the results of the following lemma implicitly utilise this action.

Lemma 3.3. *For all $\lambda \in \mathfrak{a}^*$, $\pi_1, \pi_2 \in \widehat{K}$, $g \in G$, $X, Y \in \mathfrak{g}$*

$$X\Phi_{\lambda, \pi_1, \pi_2}(g) = \sum_{\pi \in \widehat{K}} \rho_{\lambda, \pi_1, \pi}(X)\Phi_{\lambda, \pi, \pi_2}(g), \quad (3.25)$$

$$XY\Phi_{\lambda, \pi_1, \pi_2}(g) = \sum_{\pi \in \widehat{K}} \sum_{\eta \in \widehat{K}} \rho_{\lambda, \pi_1, \pi}(Y)\rho_{\lambda, \pi, \eta}(X)\Phi_{\lambda, \eta, \pi_2}(g). \quad (3.26)$$

Proof. For all $1 \leq i, k \leq d_{\pi_1}$, $1 \leq j, l \leq d_{\pi_2}$, using (3.13) and (3.12) we have

$$\begin{aligned} X\Phi_{\lambda, \pi_1, \pi_2}^{(i,j),(k,l)}(g) &= \left. \frac{d}{dt} \Phi_{\lambda, \pi_1, \pi_2}^{(i,j),(k,l)}(g \exp(tX)) \right|_{t=0} \\ &= \left. \frac{d}{dt} \langle \xi_\lambda(\exp(tX)) f_{e_{\pi_1}^i, e_{\pi_1}^k}^{\pi_1}, \xi_\lambda(g^{-1}) f_{e_{\pi_2}^j, e_{\pi_2}^l}^{\pi_2} \rangle \right|_{t=0} \\ &= \langle d\xi_\lambda(X) f_{e_{\pi_1}^i, e_{\pi_1}^k}^{\pi_1}, \xi_\lambda(g^{-1}) f_{e_{\pi_2}^j, e_{\pi_2}^l}^{\pi_2} \rangle \end{aligned}$$

and the first result follows by similar computations to those that feature in the proof of Theorem 3.3. The second result is established by iterating this argument. \square

The precise expressions for the coefficients $\rho_{\lambda, \pi_1, \pi}(X)$, for $X \in \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ are given in Lemma 3.2.

It is clear from the argument in the proof that for all $X, Y, Z \in \mathfrak{g}$ the mappings $g \rightarrow X\Phi_{\lambda, \pi_1, \pi_2}(g)$ and $g \rightarrow YZ\Phi_{\lambda, \pi_1, \pi_2}(g)$ are bounded and continuous.

4 Infinite Matrices and the Eisenstein Transform

4.1 The Exponential of an Infinite Matrix

In the next section we will need the exponential of an infinite matrix that is indexed by $\widehat{K} \times \widehat{K}$. First we introduce an identity matrix $\delta_{\pi_1, \pi_2} = \begin{cases} I & \text{if } \pi_1 = \pi_2 \\ 0 & \text{if } \pi_1 \neq \pi_2 \end{cases}$.

Let $\Upsilon(t) = (\Upsilon_{\pi_1, \pi_2}(t), \pi_1, \pi_2 \in \widehat{K})$ be a family of infinite matrices (acting on $V(K)$) indexed by $t \geq 0$ which have the following properties:

E(i) $\Upsilon(s+t) = \Upsilon(s)\Upsilon(t)$ for all $s, t \geq 0$,

E(ii) $\Upsilon_{\pi_1, \pi_2}(0) = \delta_{\pi_1, \pi_2}$,

E(iii) The mapping $t \rightarrow \Upsilon_{\pi_1, \pi_2}(t)$ is continuous for all $\pi_1, \pi_2 \in \widehat{K}$.

E(iv) The mapping $t \rightarrow \Upsilon_{\pi_1, \pi_2}(t)$ is differentiable at $t = 0$ and there exists an infinite matrix Θ so that $\frac{d}{dt} \Upsilon_{\pi_1, \pi_2}(t)|_{t=0} = \Theta_{\pi_1, \pi_2}$ for all $\pi_1, \pi_2 \in \widehat{K}$.

In this case we write $\Upsilon(t) := \text{Exp}(t\Theta)$ and call Exp an “infinite matrix exponential”. It clearly coincides with the usual exponential for finite dimensional matrices. We also write $A := \text{Exp}(\Theta)$ whenever $A = \Upsilon(1)$ for some $(\Upsilon(t), t \geq 0)$ satisfying (E(i)) to (E(iv)).

Note that for E(iii) and E(iv) (and also in calculations to follow) we assume that some fixed norm is chosen on each space of $d_{\pi_1} d_{\pi_2} \times d_{\pi_1} d_{\pi_2}$ matrices.

4.2 The Eisenstein Transform

We generalise the spherical transform of [13] and for each $\lambda \in \mathfrak{a}^*$ we introduce the *Eisenstein transform* $\widehat{\mu}_\lambda$ of a Borel probability measure μ to be the matrix valued integral defined for each $\pi, \pi_2 \in \widehat{K}$ by:

$$\widehat{\mu}_{\lambda, \pi_1, \pi_2} := \int_G \Phi_{\lambda, \pi_1, \pi_2}^\#(g) \mu(dg) \quad (4.27)$$

for all $\lambda \in \mathfrak{a}^*$. This is related in an obvious way to the group Fourier transform defined in section 8.4. Now let μ_1 and μ_2 be Borel probability measures on G and define their convolution $\mu_1 * \mu_2$ in the usual way:

$$\int_G f(g) (\mu_1 * \mu_2)(dg) = \int_G \int_G f(gh) \mu_1(dg) \mu_2(dh), \quad (4.28)$$

where f is an arbitrary bounded Borel measurable function on G . Before we investigate the Eisenstein transform of the convolution of two measures, we state a useful technical result.

Lemma 4.1. *Let U be a unitary representation of a Lie group Γ acting on a complex separable Hilbert space H and let $(e_n, n \in \mathbb{N})$ be a complete orthonormal basis for H . Then for all $x, y \in H$ the infinite series $\sum_{n=1}^\infty \langle U(g)x, e_n \rangle \langle e_n, U(h)y \rangle$ converges uniformly in g and h to $\langle U(g)x, U(h)y \rangle$.*

Proof. The result follows from unitarity and the proof of the usual Parseval identity in Hilbert space. \square

Theorem 4.1. For each $\lambda \in \mathfrak{a}^*$ (in the sense of matrix multiplication)

$$(\widehat{\mu^{(1)} * \mu^{(2)}})_\lambda = \widehat{\mu_\lambda^{(1)}} \widehat{\mu_\lambda^{(2)}}, \quad (4.29)$$

Proof. From (3.17) and (4.28) we deduce that for each $\pi_1, \pi_2 \in \widehat{K}$, $\lambda \in \mathfrak{a}^*$,

$$\begin{aligned} (\widehat{\mu^{(1)} * \mu^{(2)}})_\lambda &= \int_G \int_G \Phi_{\lambda, \pi_1, \pi_2}^\#(gh) \mu_1(dg) \mu_2(dh) \\ &= \int_G \int_G \left(\sum_{\pi \in \widehat{K}} \Phi_{\lambda, \pi_1, \pi}^\#(g) \Phi_{\lambda, \pi, \pi_2}^\#(h) \right) \mu_1(dg) \mu_2(dh). \end{aligned}$$

The result follows on interchanging the infinite sum and the integrals. This is justified as follows. Let $S \subseteq \widehat{K}$ and define for all $\lambda \in \mathfrak{a}^*$, $g, h \in G$

$$M_{\lambda, \pi_1, \pi_2}^S(g, h) := \sum_{\pi \in S} \Phi_{\lambda, \pi_1, \pi}^\#(g) \Phi_{\lambda, \pi, \pi_2}^\#(h).$$

Then by Theorem 3.3

$$\begin{aligned} M_{\lambda, \pi_1, \pi_2}^S(g, h) &= \sum_{\pi \in \widehat{K}} \Phi_{\lambda, \pi_1, \pi}^\#(g) \Phi_{\lambda, \pi, \pi_2}^\#(h) - \sum_{\pi \in S^c} \Phi_{\lambda, \pi_1, \pi}^\#(g) \Phi_{\lambda, \pi, \pi_2}^\#(h) \\ &= \Phi_{\lambda, \pi_1, \pi_2}^\#(gh) - M_{\lambda, \pi_1, \pi_2}^{S^c}(g, h). \end{aligned}$$

By the construction in the proof of Theorem 3.3 and Lemma 4.1 we can assert that given any $\epsilon > 0$ there exists a finite subset S_0 of \widehat{K} so that if $S \subseteq \widehat{K}$ is any other finite subset with $S_0 \subset S$ then $\sup_{g, h \in G} \|M_{\lambda, \pi_1, \pi_2}^{S^c}(\sigma, \tau)\| < \epsilon$. But then by Theorem 3.2

$$\begin{aligned} \sup_{g, h \in G} \|M_{\lambda, \pi_1, \pi_2}^S(g, h)\| &\leq \sup_{g, h \in G} \|\Phi_{\lambda, \pi_1, \pi_2}^\#(gh)\| + \sup_{g, h \in G} \|M_{\lambda, \pi_1, \pi_2}^{S^c}(g, h)\| \\ &\leq C_{\pi_1, \pi_2} d_{\pi_1} d_{\pi_2} + \epsilon, \end{aligned}$$

where $C_{\pi_1, \pi_2} \geq 0$ depends on the choice of matrix norm. The right hand side of the inequality is constant and thus ν -integrable and so dominated convergence can be applied to interchange the sum and the integral. \square

Remark. It is a difficult problem to find conditions for the injectivity of the mapping $\mu \rightarrow \widehat{\mu}$ beyond the known result when μ is K -bi-invariant (see Theorem 4.1 in [13]). Indeed this will not be true even in the case where μ has an L^2 -density. Some progress on the problem when μ has a C^∞ density

can be found in the generalised Paley Wiener theorem that is presented as Theorem 4.1 in [7].

For an injective Fourier transform on $L^2(G)$, one needs to analyse the discrete series and all generalized principal series, that is representations of the form $(\xi \otimes \lambda \otimes 1) \uparrow_Q^G$, with Q a cuspidal parabolic subgroup of G , ξ a discrete series rep of M_Q , and λ a unitary character of A_Q . In this paper, we have only dealt with the case where Q is minimal. We defer investigation of the other series of representations to future work. We would like to thank Erik van den Ban for useful discussions on this point [32].

5 The Extended Gangolli-Lévy Khintchine Formula

In this section we will need to make sense of expressions of the form $\mathcal{L}\Phi_{\lambda,\pi_1,\pi_2}$. Note that the matrix elements of $\Phi_{\lambda,\pi_1,\pi_2}$ do not necessarily vanish at infinity. We use the facts that the generalised Eisenstein integrals are bounded and have second order bounded derivatives (in the sense of Lie algebra actions) and that $(T_t, t \geq 0)$ extends to a locally uniformly continuous semigroup on the space $C_b(G)$ of bounded continuous functions on G (see Lemma 3.1 in [28]). Now define the space $C_b^2(G)$ in exactly the same way as $C^2(G)$ but with $C_0(G)$ replaced by $C_b(G)$ wherever the first space appears within the definition. Then \mathcal{L} extends to a linear operator in $C_b(G)$ with $C_b^2(G) \subseteq \text{Dom}(\mathcal{L})$ and $\mathcal{L}\Phi_{\lambda,\pi_1,\pi_2}$ has meaning as the $d_{\pi_1}d_{\pi_2} \times d_{\pi_1}d_{\pi_2}$ matrix with entries $\mathcal{L}\Phi_{\lambda,\pi_1,\pi_2}^{(i,j),(k,l)}$.

We return to the study of Lévy processes. Instead of (2.8), we will find it more profitable to consider

$$\alpha_{\lambda,\pi_1,\pi_2}(t) := \mathbb{E}(\Phi_{\lambda,\pi_1,\pi_2}^\#(Z(t))) = \int_G \Phi_{\lambda,\pi_1,\pi_2}(g^{-1})\mu_t(dg), \quad (5.30)$$

for all $\lambda \in \mathfrak{a}^*$, $\pi_1, \pi_2 \in \widehat{K}$, $t \geq 0$ as an infinite matrix. Let us write this matrix as $\alpha_\lambda(t)$. Our goal will be to find another infinite matrix ψ_λ with generic matrix entry $\psi_{\lambda,\pi_1,\pi_2}$ which has the property that for all $t \geq 0$:

$$\alpha_\lambda(t) = \text{Exp}(t\psi_\lambda). \quad (5.31)$$

Theorem 5.1. *Let $(\mu_t, t \geq 0)$ be the convolution semigroup of probability measures generated by \mathcal{L} . Then (5.31) holds. Furthermore for each $\lambda \in \mathfrak{a}^*$*

$$\psi_\lambda = -b^i \rho_\lambda(X_i) + a^{ij} \rho_\lambda(X_i) \rho_\lambda(X_j) + \eta_\lambda, \quad (5.32)$$

where η_λ is the matrix whose diagonal entries are

$$\int_G (\Phi_{\lambda,\pi,\pi}^\#(\tau) - 1 + x^i(\tau) \rho_{\lambda,\pi,\pi}(X_i)) \nu(d\tau)$$

and off-diagonal entries are

$$\int_G (\Phi_{\lambda,\pi_1,\pi_2}^\#(\tau) + x^i(\tau) \rho_{\lambda,\pi_1,\pi_2}(X_i)) \nu(d\tau).$$

Proof. We first check that the integrals defining the entries of the infinite matrix η_λ in (5.32) are finite. To establish this we must show that the integrand is $O(\sum_{i=1}^n x_i^2)$ in a canonical co-ordinate neighbourhood V of e . Note that we may choose x_1, \dots, x_n so that $x(g^{-1}) = -x(g)$ for all $g \in V$. For each $g \in V$, we write $g = \exp(X)$ where $X \in \mathfrak{g}$ and we use the fact that for each $\lambda \in \mathfrak{a}^*$, $\xi_\lambda(\exp(X)) = \exp(d\pi(\xi_\lambda(X)))$ where the second \exp is understood in the sense of functional calculus for skew-adjoint operators. Now using a Taylor expansion, (3.12), (3.24) and Lemma 3.3 we find that for all $\pi_1, \pi_2 \in \widehat{K}$, $g \in V$:

$$\begin{aligned} \Phi_{\lambda,\pi_1,\pi_2}(g^{-1}) &= \delta_{\pi_1,\pi_2} - \sum_{i=1}^n x_i(g) \rho_{\lambda,\pi_1,\pi_2}(X_i) \\ &+ \sum_{i,j=1}^n x_i(g) x_j(g) \sum_{\pi,\eta \in \widehat{K}} \rho_{\lambda,\pi_1,\pi}(X_i) \rho_{\lambda,\pi,\eta}(X_j) \Phi_{\lambda,\eta,\pi_2}(g'), \end{aligned}$$

where $g' \in V$, from which the required integrability follows.

To prove the theorem we need to show that E(i) to E(iv) are satisfied for $(\alpha_\lambda(t), t \geq 0)$. E(i) follows from Theorem 4.1 since we have $\mu_{t+s} = \mu_t * \mu_s$ for all $s, t \geq 0$, E(ii) holds since $\mu_0 = \delta_e$ and $\Phi_{\lambda,\pi_1,\pi_2}(e) = \delta_{\pi_1,\pi_2}$ for all $\pi_1, \pi_2 \in \widehat{K}$. E(iii) is clear from the fact that $t \rightarrow \mu_t$ is weakly continuous. To establish E(iv) we argue as in the Remark on page 23 of [24] (see also equation (8) in [25]) to see that

$$\left. \frac{d}{dt} \int_G \Phi_\lambda^\#(g) \mu_t(dg) \right|_{t=0} = \mathcal{L} \Phi_\lambda^\#(e) = \psi_\lambda,$$

where the last identity is obtained by explicit calculation. \square

Corollary 5.1. *Let μ be an infinitely divisible probability measure on G . Then for all $\lambda \in \mathfrak{a}^*$ we have*

$$\int_G \Phi_\lambda^\#(g) \mu(dg) = \text{Exp}(\psi_\lambda),$$

where the matrix ψ_λ is as in Theorem 5.1.

Proof. From the remarks at the beginning of section 6 in [11] we see that any connected semisimple Lie group has the “embedding property” that $\mu = \mu_1$ for some convolution semigroup of probability measures $(\mu_t, t \geq 0)$ on G . The result then follows from Theorem 5.1. \square

We have a partial converse to this result, but the last part should be read in conjunction with the Remark at the end of section 4.

Proposition 5.1. *Let $\psi_\lambda^{(b,a,\nu)}$ be an infinite matrix of the form given in Theorem 5.1 (where we emphasise the dependence on the triple (b, a, ν) comprising a vector, a non-negative definite matrix a and a Lévy measure ν on $G - \{e\}$.) Then there is an infinitely divisible probability measure μ on G so that*

$$\int_G \Phi_\lambda^\#(g) \mu(dg) = \text{Exp}(\psi_\lambda) \quad (5.33)$$

for all $\lambda \in \mathfrak{a}^*$. Furthermore if μ is such that the Eisenstein transform $\mu \rightarrow \widehat{\mu}$ is injective then μ is uniquely defined by (5.33).

Proof. Using the triple (b, a, ν) we construct a linear operator of the form (2.6) acting on $C^2(G)$. By Theorem 2.1 this operator extends to be the infinitesimal generator of a unique convolution semigroup of probability measures $(\mu_t, t \geq 0)$. Let $(T_t, t \geq 0)$ be the corresponding semigroup of convolution operators. Then by Theorem 5.1 we have for all $t \geq 0$,

$$T_t \Phi_{\lambda, \pi_1, \pi_2}(e) = \int_G \Phi_{\lambda, \pi_1, \pi_2}(g^{-1}) \mu_t(dg) = \text{Exp}(t \psi_{\lambda, \pi_1, \pi_2}^{(b,a,\nu)})$$

and $\mu := \mu_1$.

Now suppose that we can find another triple (b', a', ν') so that $\psi_\lambda^{(b,a,\nu)} = \psi_\lambda^{(b',a',\nu')}$. Then by the above construction we can find a convolution semigroup $(\mu'_t, t \geq 0)$ for which $\int_G \Phi_{\lambda, \pi_1, \pi_2}(g^{-1}) \mu_t(dg) = \int_G \Phi_{\lambda, \pi_1, \pi_2}(g^{-1}) \mu'_t(dg)$ and so if the Eisenstein transform is injective then $\mu_t = \mu'_t$ for all $t \geq 0$. \square

6 Extension to Symmetric Spaces

Let μ be a right K -invariant Borel probability measure on G so that $\mu(Ak) = \mu(A)$ for all $k \in K$ and all $A \in \mathcal{B}(G)$. We now consider the Eisenstein transform $\widehat{\mu}_\lambda$ of μ as defined in (4.27). Let δ denote the trivial representation of K .

Theorem 6.1. *If μ is a right K -invariant probability measure on G then for all $\lambda \in \mathfrak{a}^*$, $\pi \in \widehat{K}$, $\widehat{\mu}_{\lambda, \pi_1, \pi} = 0$ except when $\pi_1 = \delta$.*

Proof. By (4.27), (3.12) and using the right K -invariance of μ we obtain for all $\pi_1 \in \widehat{K}$, $u_1, v_1 \in V_{\pi_1}$, $u, v \in V_{\pi}$,

$$\begin{aligned}
\langle \widehat{\mu}_{\lambda, \pi_1, \pi}(u_1 \otimes u^*), v_1 \otimes v^* \rangle &= \int_G \langle \Phi_{\lambda, \pi_1, \pi}^{\#}(g)(u_1 \otimes u^*), v_1 \otimes v^* \rangle \mu(dg) \\
&= \int_G \langle \xi_{\lambda}(g) f_{u_1, v_1}^{\pi_1}, f_{u, v}^{\pi} \rangle \mu(dg) \\
&= \int_K \int_G \langle \xi_{\lambda}(g) f_{u_1, v_1}^{\pi_1}, f_{u, v}^{\pi} \rangle \mu(dg) dk \\
&= \int_K \int_G \langle \xi_{\lambda}(gk) f_{u_1, v_1}^{\pi_1}, f_{u, v}^{\pi} \rangle \mu(dg) dk \\
&= \int_G \left\langle \left(\int_K \xi_{\lambda}(k) f_{u_1, v_1}^{\pi_1} dk \right), \xi(g^{-1}) f_{u, v}^{\pi} \right\rangle \mu(dg)
\end{aligned}$$

by Fubini's theorem. However by Peter-Weyl theory, for each $l \in K$

$$\int_K \xi_{\lambda}(k) f_{u_1, v_1}^{\pi_1}(l) dk = \int_K \langle \pi_1(l) \pi_1(k) u_1, v_1 \rangle dk = \int_K \langle \pi_1(k) u_1, v_1 \rangle dk = 0$$

unless $\pi_1 = \delta$, and the result follows. \square

Now let μ be right K -invariant. Then by Theorem 6.1, the non-trivial values of $\widehat{\mu}_{\lambda}$ are given by

$$\widehat{\mu}_{\lambda, \pi} := \widehat{\mu}_{\lambda, \delta, \pi} = \int_G \Phi_{\lambda, \delta, \pi}^{\#}(g) \mu(dg) = \int_G \Phi_{\lambda, \pi}^{\#}(g) \mu(dg),$$

where we recall that $\Phi_{\lambda, \pi}$ is the Eisenstein integral (2.7). For the statement of the next theorem we will also require the (Harish-Chandra) spherical function, $\Phi_{\lambda} := \Phi_{\lambda, \delta}$.

Now let $M := G/K$ be the Riemannian globally symmetric space whose points are the left cosets $\{gK, g \in G\}$ and let $\natural : G \rightarrow G/K$ be the canonical surjection. Let $C_K(G)$ denote the closed subspace of $C_0(G)$ comprising those functions that are right K -invariant and observe that there is a natural isometric isomorphism between $C_0(G/K)$ and $C_K(G)$ given by the prescription $f \rightarrow f \circ \natural$ for each $f \in C_0(G/K)$. Let μ be a right K -invariant Borel probability measure on G . Then $\tilde{\mu} = \mu \circ \natural^{-1}$ defines a Borel probability measure on M . We define the *Eisenstein transform* of $\tilde{\mu}$ to be $\widehat{\mu}$ and note that this is well-defined as the Eisenstein transform of μ is invariant under the right K action on the measure.

Now let $(\mu_t, t \geq 0)$ be a weakly continuous convolution semigroup of right invariant probability measures on G where instead of requiring $\mu_0 = \delta_e$ (which is not right invariant) we take μ_0 to be normalised Haar measure on

K (equivalently $\mu_0(\cdot) = \int_K \delta_e(\cdot k) dk$, in the weak sense.) This may arise as the law of a G -valued Lévy process Z with right K -invariant laws for which $Z(0)$ is uniformly distributed on K and is independent of $Z(t)$ for $t > 0$. The measures $(\mu_t, t \geq 0)$ induce a C_0 contraction semigroup $(T_t, t \geq 0)$ having generator \mathcal{L} on $C_0(G/K)$ in the usual manner, i.e. $T_t f = f * \mu_t$ for all $f \in C_0(G/K), t \geq 0$. We induce a C_0 semigroup of operators $(\tilde{T}_t, t \geq 0)$ on $C_0(G/K)$ by the prescription

$$(\tilde{T}_t f) \circ \mathfrak{h} = T_t(f \circ \mathfrak{h}),$$

for all $t \geq 0, f \in C_0(G/K)$. Let \tilde{L} be the infinitesimal generator of $(\tilde{T}_t, t \geq 0)$ then $(\tilde{L}f) \circ \mathfrak{h} = L(f \circ \mathfrak{h})$ for all $f \in \text{Dom}(\tilde{L}) := \{f \in C_0(G/K); f \circ \mathfrak{h} \in \text{Dom}(L)\}$. We define $C_2(G/K) := \{f \in C_0(G/K); f \circ \mathfrak{h} \in C_2(G)\}$. Then $C_2(G/K)$ is dense in $C_0(G/K)$ and $C_2(G/K) \subseteq \text{Dom}(\tilde{L})$. By restricting the matrix calculation in the proof of Theorem 5.1 to preserve the row containing Eisenstein integrals we deduce the following which can be seen as a generalised Lévy-Khintchine formula on the symmetric space M . In the following, for each $\lambda \in \mathfrak{a}^*, t \geq 0$ we regard $\hat{\mu}_t(\lambda)$ and ψ_λ as row-vectors where the value of the former at $\pi \in \hat{K}$ is $\hat{\mu}_t(\lambda, \pi) := \hat{\mu}_t(\lambda, \delta, \pi)$.

Theorem 6.2. *Let $(\mu_t, t \geq 0)$ be a right K -invariant convolution semigroup of probability measures on G . Then for all $\lambda \in \mathfrak{a}^*, t > 0$ we have that $\hat{\mu}_t(\lambda)$ is the row vector of $\text{Exp}(t\psi_\lambda)$ whose entries are of the form (δ, π) for $\pi \in \hat{K}$. Furthermore for all $\pi \in \hat{K}$*

$$\left. \frac{d}{dt} \hat{\mu}_t(\lambda, \pi) \right|_{t=0} = \psi_{\lambda, \pi} \quad (6.34)$$

where

$$\begin{aligned} \psi_{\lambda, \pi} &= \psi_{\lambda, \delta, \pi} \\ &= -b^i \rho_{\lambda, \delta, \pi}(X_i) + \sum_{\eta \in \hat{K}} a^{ij} \rho_{\lambda, \delta, \eta}(X_i) \rho_{\lambda, \eta, \pi}(X_j) + \eta_{\lambda, \pi}. \end{aligned}$$

Here $\eta_{\lambda, \pi}$ is the row vector whose (δ, δ) entry is

$$\int_G (\Phi_\lambda^\#(\tau) - 1) \nu(d\tau)$$

and (δ, π) entries (for $\pi \neq \delta$) are

$$\int_G (\Phi_{\lambda, \pi}^\#(\tau) + x^i(\tau) \rho_{\lambda, \delta, \pi}(X_i)) \nu(d\tau).$$

Proof. This follows from the proof of Theorem 5.1. \square

In particular, when μ_t is K -bi-invariant for all $t \geq 0$ then we recover the characteristic exponent in Gangolli's Lévy-Khintchine formula (for which see [13], [25], [2]) as $\psi_{\lambda, \delta}$.

7 Example: $SU(1, 1)$ and Hyperbolic Space

The group $SU(1, 1)$ is by definition the set of 2×2 complex matrices of the form $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ for which $|a|^2 - |b|^2 = 1$. The group acts on the unit disc \mathcal{D} in \mathbb{C}^2 by fractional linear transformations.

It is worth noting that the Cayley transform $z \mapsto \frac{z-i}{z+i}$ takes the upper half plane to \mathcal{D} , and that conjugation with the corresponding matrix $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ gives an isomorphism between $SU(1, 1)$ and $SL(2, \mathbb{R})$. Thus the representation theory of $SU(1, 1)$, which we are about to describe, is the representation theory of $SL(2, \mathbb{R})$ in another guise.

The Lie algebra $\mathfrak{su}(1, 1)$ is given by the 2×2 complex matrices of the form $\begin{pmatrix} ix & b \\ \bar{b} & -ix \end{pmatrix}$ where $x \in \mathbb{R}$ and $b \in \mathbb{C}$.

The Cartan involution is $\theta(A) = -\bar{A}^t$ for $A \in \mathfrak{su}(1, 1)$. The fixed points of θ are the real linear span \mathfrak{k} of the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ which corresponds to the subgroup $SO(2) = \left\{ k(\psi) = \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix} : \psi \in [0, 2\pi) \right\}$.

Letting \mathfrak{p} be the linear span of $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\}$, we have $\mathfrak{su}(1, 1) = \mathfrak{k} \oplus \mathfrak{p}$.

We choose our maximal abelian subalgebra of \mathfrak{p} to be \mathfrak{a} which is the linear span of $\left\{ H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

Notice that for each $H \in \mathfrak{a}$, $\exp(tH) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ for $t \in \mathbb{R}$, and that these matrices form the group A .

A straightforward calculation shows that the root spaces for \mathfrak{a} are spanned by $X_\alpha = \begin{pmatrix} i & -i \\ i & -i \end{pmatrix}$ with root $\alpha(H) = 2$, and $X_{-\alpha} = \begin{pmatrix} -i & -i \\ i & i \end{pmatrix}$ with root $-\alpha$.

The weight ρ is equal to $\frac{\alpha}{2}$, so that $\rho(H) = 1$.

One calculates that $\theta X_\alpha = X_{-\alpha}$, that $X_\alpha + \theta X_\alpha = 2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ belongs to \mathfrak{k} , and that $X_\alpha - \theta X_\alpha = 2 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ belongs to \mathfrak{p} .

A simple calculation shows that $[X_\alpha, \theta X_\alpha] = 4H$.

The Iwasawa decomposition for $SU(1, 1)$ is $G = NAK$, where K and A are as above, and $N = \{\exp tX_\alpha : t \in \mathbb{R}\}$ is the subgroup $\left\{ \begin{pmatrix} 1+it & -it \\ it & 1-it \end{pmatrix} : t \in \mathbb{R} \right\}$.

The subgroup M of K which fixes H is $\{I, -I\}$. There are two irreducible representations of M , the trivial representation $\sigma_+(m) = 1$ for $m = \pm I$, and the self representation $\sigma_-(\pm I) = \pm 1$.

We now describe the principal series representations of $SU(1, 1)$. As indicated in the appendix, they are indexed by the characters of A , specified by the real numbers λ , corresponding to the characters $\exp(tH) \mapsto e^{i\lambda t}$.

For each $\lambda \in \mathbb{R}$, we define ξ_λ acting on $L^2(SO(2))$ by

$$(\xi_\lambda(g)f)(l) = e^{(i\lambda+1)(A(lg))} f(u(lg)),$$

where $f \in L^2(SO(2))$, $l \in SO(2)$, $g = n \exp(A(g))u(g) \in SU(1, 1)$.

The two subspaces of $L^2(SO(2))^\pm$ consisting (respectively) of the even functions $\{f : f(-k) = f(k)\}$ and the odd functions $\{f : f(-k) = -f(k)\}$ on the circle then each carry an irreducible representation ξ_λ^+ and ξ_λ^- , corresponding to the two irreducible representations of M .

Now we can decompose $\mathcal{H} = L^2(SO(2))$ into K -types, using the Fourier transform: that is, for all $n \in \mathbb{Z}$ we let \mathcal{H}_n be the linear span of the character $\theta \mapsto e^{in\theta}$; these spaces are all one-dimensional for $SU(1, 1)$. We have $\mathcal{H}^\pm = \bigoplus_n \mathcal{H}_n$, the sum being over even integers for \mathcal{H}^+ and over odd integers for \mathcal{H}^- .

The generalised Eisenstein integrals are indexed by λ together with a pair (n', n) in $\mathbb{Z} \times \mathbb{Z}$.

$$\Phi_{\lambda, n', n}(g) = \frac{1}{2\pi} \int_0^{2\pi} e^{(i\lambda+1)(A(k(\theta)g))} e^{in'(u(k(\theta)g)) - in\theta} d\theta. \quad (7.35)$$

The usual Eisenstein integrals are recovered when $n' = 0$.

Now let us pass to hyperbolic space \mathcal{D} , which we may realise as $G/K = SU(1, 1)/SO(2)$ via $M.z = \frac{az+b}{bz+\bar{a}}$, where of course M is the matrix $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$.

Notice that the stabiliser of the point $z = 0$ is precisely the group K . Thus the mapping $M \mapsto \frac{b}{a}$ is a one-one correspondence between G/K and \mathcal{D} . Note that $\exp(tH)$ maps to $\tanh(t)$ through this correspondence. As shown by Helgason [16] (see section 4 of the Introduction, pp. 29-72) the action of

$SU(1, 1)$ is isometric for the usual Riemannian structure of the Poincaré disc. The boundary $\mathcal{B} = \partial\mathcal{D}$ is identified as $K = SO(2)$. Furthermore, consider the expression $A(k(\cdot)g)$ in the exponent of the integrand of formula (7.35). As the A part of g coincides with the A part of gk for all $g \in G, k \in K$, this expression passes to the quotient and defines an element of A as a function of k and gK . Writing this element of A in the form $\exp(tH)$, let us denote $t = \langle gK, k \rangle$. This is consistent with Helgason's notation $\langle z, b \rangle$ for $z \in \mathcal{D}, b \in \mathcal{B}$.

Now we can express the image of the Eisenstein integrals (7.35) under the quotient map from G to \mathcal{D} , as

$$\Phi_{\lambda,n}(z) := \Phi_{\lambda,0,n}(z) = \int_{\mathcal{B}} e^{(i\lambda+1)\langle z,b \rangle} b^{-n} db$$

It follows readily from this that if $z = e^{i\theta}r$, with $r \geq 0$, then

$$\Phi_{\lambda,n}(z) = e^{in\theta} \Phi_{\lambda,n}(r).$$

Furthermore, the following formula is derived in [16], p.60:

$$\Phi_{\lambda,n}(r) = r^{|n|} \frac{\Gamma(|n| + \nu)}{\Gamma(\nu)|n|!} {}_2F_1 \left(\nu, 1 - \nu; |n| + 1; \frac{r^2}{r^2 - 1} \right), \quad (7.36)$$

where $\nu = \frac{1}{2}(1 + i\lambda)$ and ${}_2F_1$ is the Gauss hypergeometric function.

Notice that $\Phi_{\lambda,0}(z) = \int_{\mathcal{B}} e^{(i\lambda+1)\langle z,b \rangle} db$ is the usual spherical function, denoted ϕ_λ by Helgason.

Now if we take the Fourier transform of $f \in L^1(D)$, we obtain

$$\begin{aligned} \int_{\mathcal{D}} f(z) \Phi_{\lambda,n}(z) dz &= \int_{\mathcal{D}} f(z) \int_{\mathcal{B}} e^{(i\lambda+1)\langle z,b \rangle} b^{-n} db dz \\ &= \int_{\mathcal{B}} \int_{\mathcal{D}} f(z) e^{(i\lambda+1)\langle z,b \rangle} dz b^{-n} db \\ &= \int_{\mathcal{B}} \tilde{f}(\lambda, b) b^{-n} db \\ &= \widehat{\tilde{f}}(\lambda, \cdot)(n), \end{aligned}$$

where, in Helgason's notation, $\tilde{f}(\lambda, b) = \int_{\mathcal{D}} f(z) e^{(i\lambda+1)\langle z,b \rangle} dz$ is the "Fourier transform" on \mathcal{D} .

Notice that Helgason's Fourier transform is a function of (λ, k) , for k in the circle group. The Helgason Fourier transform can be recovered from the group Fourier transform by use of the usual Fourier series, specifically:

$$\tilde{f}(\lambda, b) = \sum_{n \in \mathbb{Z}} \left(\int_{\mathcal{D}} f(z) \phi_{\lambda,n}(z) dz \right) b^n.$$

With this notation, the characteristic exponent in (6.34) has the form

$$\psi_{\lambda,n} = -b^i \rho_{\lambda,0,n}(X_i) + \sum_{k \in \mathbb{Z}} a^{i,j} \rho_{\lambda,0,k}(X_i) \rho_{\lambda,k,n}(X_j) + \eta_{\lambda,n},$$

for each $\lambda \in \mathbb{R}$, $n \in \mathbb{Z}$ and where we note that $i, j \in \{1, 2\}$.

We can calculate $\rho_{\lambda,0,k}$ as follows. The tangent space is identified with \mathfrak{p} , and we choose as orthonormal basis $X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

Then $\exp tX_1 = \begin{pmatrix} \cosh \frac{t}{\sqrt{2}} & \sinh \frac{t}{\sqrt{2}} \\ \sinh \frac{t}{\sqrt{2}} & \cosh \frac{t}{\sqrt{2}} \end{pmatrix}$ which maps under the quotient map to $\tanh \frac{t}{\sqrt{2}}$ in \mathcal{D} , and $\exp tX_2 = \begin{pmatrix} \cosh \frac{t}{\sqrt{2}} & i \sinh \frac{t}{\sqrt{2}} \\ -i \sinh \frac{t}{\sqrt{2}} & \cosh \frac{t}{\sqrt{2}} \end{pmatrix}$ which maps under the quotient map to $i \tanh \frac{t}{\sqrt{2}}$ in \mathcal{D} . From the above, it follows that $\rho_{\lambda,0,n}(X_2) = i^n \rho_{\lambda,0,n}(X_1)$, and by definition, $\rho_{\lambda,0,n}(X_1) = \frac{d}{dt} \Phi_{\lambda,n}(\tanh \frac{t}{\sqrt{2}})|_{t=0}$. By equation (7.36),

$$\Phi_{\lambda,n}(\tanh \frac{t}{\sqrt{2}}) = (\tanh \frac{t}{\sqrt{2}})^{|n|} \frac{\Gamma(|n| + \nu)}{\Gamma(\nu)|n!} {}_2F_1 \left(\nu, 1 - \nu; |n| + 1; \sinh^2 \frac{t}{\sqrt{2}} \right).$$

A simple computation involving the hypergeometric series shows that the derivative of this function evaluated at $t = 0$ is $\frac{\nu(1-\nu)}{2} = \frac{1}{8}(1 + \lambda^2)$ if $n = 0$, $\frac{\nu}{2} = \frac{1}{4}(1 + i\lambda)$ if $n = \pm 1$ and zero otherwise.

Thus $\rho_{\lambda,0,0}(X_1) = \rho_{\lambda,0,0}(X_2) = \frac{1}{4}(1 + \lambda^2)$, $\rho_{\lambda,0,n}(X_1) = \frac{1}{4}(1 + i\lambda)$ if $n = \pm 1$, and $\rho_{\lambda,0,n}(X_2) = \frac{\pm i}{4}(1 + i\lambda)$ if $n = \pm 1$. Both coefficients are zero for $n \neq \pm 1, 0$.

These formulae are the expressions for hyperbolic space of the infinitesimal action of Lemma 3.2.

We have $\eta_{\lambda,0} = \int_{\mathcal{D}} (\phi(z) - 1) \tilde{\nu}(dz)$, and for $n \neq 0$

$$\eta_{\lambda,n} = \int_{\mathcal{D}} (\Phi_{\lambda,n}(z) + \tilde{x}_i(z) \rho_{\lambda,0,n}(X^i)) \tilde{\nu}(dz),$$

where $\tilde{\nu}$ is a Lévy measure on D defined in terms of the given Lévy measure ν on $SU(1, 1)$ by $\tilde{\nu} = \nu \circ \mathfrak{h}^{-1}$. Similarly $\tilde{x}_i(z) = x_i \circ \mathfrak{h}^{-1}$ for $i = 1, 2$. In the spherically symmetric case, the spherical function ϕ is a Legendre function and an explicit Lévy-Khintchine formula is presented in Theorem 7.4 of [15]. To recover this result, notice that for the K -bi-invariant case, we have

$$\begin{aligned} \psi_{\lambda,0} &= a \sum_{i=1,2} \rho_{\lambda,0,0}(X_i)^2 + \eta_{\lambda,0} \\ &= \frac{1}{2} a (1 + \lambda^2) + \int_{\mathcal{D}} (\phi(z) - 1) \tilde{\nu}(dz) \end{aligned}$$

for a non-negative constant a .

8 Appendix: Some Aspects of Representation Theory

In this paper, we have used the Fourier transform on a semisimple group as an essential tool. In order to make the article more self-contained and accessible, we give here a brief introduction to this subject, showing how the Mackey theory of induced representations (which derives from work of Schur) is applied to the case of semisimple Lie groups to derive the principal series, and giving some of the basic ideas about the Fourier transform for locally compact groups. A certain divergence of notation can be found in the literature: one can choose whether to write the induced representation on the left (à la Wallach [33]) or the right (à la Knapp [22]); we have chosen the latter, which means there is no g^{-1} in the formula, but means that the representation is realised on L^2 of the right coset space $H\backslash G$. One must also decide whether to write the Iwasawa decomposition as $G = KAN$ or ANK or NAK . We have chosen NAK . Each of the other choices gives slightly different formulae, although of course the representation theory is the same. Here we have aimed at a consistent choice that provides simple formulae, consistent with those of Helgason [17]. Notice that although we use right coset spaces for the representation theory of G , our choice of Iwasawa decomposition means that it is natural to represent the symmetric space G/K as **left** cosets, as we do in section 5 above.

8.1 Induced Representations

Let G be a locally compact group and H a closed subgroup. Mackey's theory shows how to start from a (continuous unitary) representation π of H acting in \mathcal{H}_π and construct a continuous unitary representation $\xi = \pi \uparrow_H^G$ of G , which generalises Schur's construction from finite groups, and has good functorial properties.

Before making the construction, we state Mackey's Lemma.

Lemma 8.1. *If G is second countable and H is closed in G , then the quotient space $H\backslash G = \{Hg : g \in G\}$ has a right action of G by $g : Hg_1 \mapsto Hg_1g^{-1}$ and there exists a Borel measure μ on $H\backslash G$ which is quasi-invariant for this action, with the property that the measure of each open set is strictly positive. Furthermore there is at least one open set having finite μ -measure.*

By Lemma 8.1 and the Radon-Nikodým Theorem, we can define an a.e.

unique function $\frac{d\mu \circ g}{d\mu} : H \backslash G \rightarrow \mathbb{R}^+$ such that for all Borel sets $E \subseteq H \backslash G$,

$$\int_E \frac{d\mu \circ g}{d\mu}(h) \mu(dh) = \mu(Eg^{-1}).$$

The construction will give ξ acting by a slight modification of the right regular representation in a space of functions on G .

Given π and \mathcal{H}_π as above, we let

$$\mathcal{H}_\xi = \{f : G \rightarrow \mathcal{H}_\pi : f(hg) = \pi(h)f(g) \text{ and } \int_{H \backslash G} \|f(Hg)\|^2 d\mu(Hg) < \infty\} \quad (8.37)$$

In order to make sense of the integral in the above definition, notice that if f and k both satisfy the invariance condition, then their inner product is constant on cosets of H in the sense that

$$\langle f(hg), k(hg) \rangle_{\mathcal{H}_\pi} = \langle f(g), k(g) \rangle_{\mathcal{H}_\pi}. \quad (8.38)$$

Thus, taking $f = k$, the integrand in (8.37), evaluated at g , is seen to depend only on the coset Hg . In fact, we may define an inner product on \mathcal{H}_ξ by $\langle f, k \rangle_{\mathcal{H}_\xi} = \int_{H \backslash G} \langle f(Hg), k(Hg) \rangle d\mu(Hg)$.

Now, we define $\xi(g)f(g_1) = f(g_1g) \left(\frac{d\mu \circ g}{d\mu}(Hg_1) \right)^{1/2}$. A simple calculation shows that ξ is unitary.

Note that each function in \mathcal{H}_ξ is determined by its values on a section of the quotient $H \backslash G$. Indeed, given a section $\gamma : H \backslash G \rightarrow G$, i.e. a measurable map such that $\gamma(Hg)$ lies in the same coset as g , we can uniquely factorise each element g of G as $g = h(g)\gamma(Hg)$ for some $h(g) \in H$. Now if f belongs to \mathcal{H}_ξ , then

$$f(g) = f(h(g)\gamma(Hg)) = \pi(h(g))f(\gamma(Hg)).$$

Actually, it is better in some cases to think of \mathcal{H}_ξ as the space $L^2(H \backslash G, d\mu) \otimes \mathcal{H}_\pi$. In this picture, the representation ξ acts on an elementary tensor $f \otimes \eta$, $f \in L^2(H \backslash G, d\mu)$ and $\eta \in \mathcal{H}_\pi$ by

$$\xi(g)f \otimes \eta = \mathcal{R}_g f \otimes \pi(h(g))\eta, \quad (8.39)$$

where \mathcal{R}_g is the natural action of G on $L^2(H \backslash G)$ given by $\rho_g f(Hg_1) = f(g_1g) \left(\frac{d\mu \circ g}{d\mu}(Hg_1) \right)^{1/2}$. The presence of the Radon-Nikodým derivative ensures that the L^2 norm is preserved.

8.2 The Principal Series

We are going to apply the theory of the previous subsection to the case where G is a semi-simple Lie group and H is a minimal parabolic subgroup of G .

We have the Iwasawa decomposition $G = NAK$ and we write $g \in G$ as $g = n(g) \exp(H(g))\tau(g)$, where $n(g) \in N$, $H(g) \in \mathfrak{a}$ and $\tau(g) \in K$.

As usual, M is the centraliser of A in K . The minimal parabolic subgroup of G associated to these choices is NAM . As M normalises both A and N , this group has the structure of a semi-direct product of N by the actions of A and M . Given $\lambda \in \mathfrak{a}^*$, we have a character of A given by $\exp(H) \mapsto e^{i\lambda(H)}$ for $H \in \mathfrak{a}$. This gives a character of NA by $(n, \exp(H)) \mapsto e^{i\lambda(H)}$. Given also an irreducible representation ν of M , we let $\pi_{\lambda, \nu}(n, \exp(H), m)\eta = e^{i\lambda(H)}\nu(m)\eta$, and this provides us with a representation of NAM in \mathcal{H}_ν .

The principal series is defined to be the family $\xi_{\lambda, \nu} = \pi_{\lambda, \nu} \uparrow_{NAM}^G$.

We give some more explicit formulae for the principal series representations. Firstly, notice that the coset space $H \backslash G$ is $NAM \backslash NAK \sim M \backslash K$. If we realise the coset space as $M \backslash K$, we have

Lemma 8.2. *The push-forward of the Haar measure of K gives an K invariant measure on $M \backslash K$. Furthermore, this measure is quasi-invariant for the action of NAK . The Radon-Nikodým derivative for the action of G is given by*

$$\frac{d\mu \circ g}{d\mu}(Mk) = e^{2\rho(H(kg))},$$

where ρ is half the sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$.

Now, using the formulae of the previous section, $\xi_{\lambda, \nu}$ acts in the space

$$\mathcal{H}_{\lambda, \nu} = \{f : G = NAK \rightarrow \mathcal{H}_\nu : f(namg) = e^{i\lambda(H)}\nu(m)f(g)\}$$

with the action given by

$$\xi_{\lambda, \nu}(g)f(g_1) = f(g_1g)e^{\rho(H(g_1g))}.$$

It is interesting to re-write this action using the notation of formula (8.39), as we then obtain an explicit representation on $L^2(M \backslash K)$. Notice that the map $g \mapsto \tau(g)$ maps nak to k , and, because of the normalising property of M , $\tau(mg) = m\tau(g)$. It follows that $g \mapsto M\tau(g)$ is a section for the quotient map $g \mapsto NAMg$.

We may now write the action of $\xi_{\lambda, \nu}(g)$ on $L^2(M \backslash K) \otimes \mathcal{H}_\nu$ as

$$\xi_{\lambda, \nu}(g)(f \otimes \eta)(l) = e^{(i\lambda + \rho)(H(lg))} \mathcal{R}_g f \otimes \nu(m(k))\eta.$$

Here, $\mathcal{R}_g f(Mk) = f(M\tau(kg))$, and we suppose that there is a section $p : M \backslash K \rightarrow K$ where each $k \in K$ can be written $k = p(Mk)m(k)$ for a suitable $m(k) \in M$.

Actually, there is a slightly more straightforward way of writing this representation. Consider the representation of G on $L^2(K)$ given by

$$\xi'_\lambda(g)(f)(l) = e^{(i\lambda+\rho)(H(lg))} f(\tau(lg)), \quad (8.40)$$

for $g \in G, l \in K$. This representation extends to $\xi'_\lambda \otimes \mathbf{1}$ acting on $L^2(K) \otimes \mathcal{H}_\nu$ trivially on the second coordinate. Now consider the subspace generated by the elementary tensors such that $\xi'_\lambda(m)f \otimes \eta = f \otimes \nu(m)\eta$ for all $m \in M$. The restriction of $\xi'_\lambda \otimes \mathbf{1}$ to this subspace is equivalent to $\xi_{\lambda,\nu}$.

Notice that $\xi'_\lambda(k)f(k_1) = f(k_1k)$ is exactly the right regular representation of K on $L^2(K)$.

In fact, see [22], for each cuspidal parabolic subgroup Q of G , there is a Q -series of representations of G ; we have only dealt with the case when Q is a minimal parabolic subgroup. Another feature that one can vary is the assumption that λ is real-valued; thus we obtain non-unitary principal series. The *Casselman subrepresentation theorem* states that every irreducible admissible representation of G appears (on the K -finite level) as a subrepresentation of some non-unitary principal series. The description of all irreducible representations of G is thus somewhat more complicated than we have described here.

8.3 K-types

Suppose that G is a locally compact group and K a compact subgroup. We are going to apply this theory to the case where G is a connected semi-simple Lie group having a finite centre and K is a maximal compact group, but for the moment, we keep the discussion general.

If ξ is a representation of G on \mathcal{H} , we want to consider the restriction $\xi|_K$. This is a representation of a compact group, which may be written as a direct sum of irreducible representations of K . Letting \hat{K} be the unitary dual of K (i.e. a maximal set of pairwise inequivalent irreducible representations of K), we may write

$$\xi|_K = \bigoplus_{\pi \in \hat{K}} m_\pi \pi, \quad (8.41)$$

where $1 \leq m_\pi \leq \infty$ is the multiplicity of π in the decomposition.

If G is a connected semi-simple Lie group having a finite centre, K is a maximal compact subgroup and ξ' is the principal series constructed in the previous section, we saw that $\xi'|_K$ is the regular representation of K on

$L^2(K)$. By the Peter-Weyl theorem, the regular representation contains each irreducible representation π of K exactly d_π times. Thus, in this example, m_π is nothing but the dimension of the representation space.

We shall continue our general discussion, under the assumption that $m_\pi < \infty$ for all $\pi \in \hat{K}$. Equation (8.41) means that there exists a family of orthogonal projections on \mathcal{H} , $\{P_i^\pi : \pi \in \hat{K}, i = 1, \dots, m_\pi\}$, each with finite range of dimension d_π , so that $P_i^\pi \xi|_{K P_i^\pi}$ is equivalent to $\pi \in \hat{K}$, for $i = 1, \dots, m_\pi$, and such that

$$\bigoplus_{\pi \in \hat{K}} \bigoplus_{i=1}^{m_\pi} P_i^\pi = \text{Id}.$$

Let $P_i^\pi \mathcal{H} = \mathcal{H}_i^\pi$. Then \mathcal{H}_i^π carries an irreducible representation of K equivalent to π for $i = 1, \dots, m_\pi$. Furthermore, this representation acts by $\pi(k)\phi = \xi(k)\phi$ for $\phi \in \mathcal{H}_i^\pi$.

This decomposition of \mathcal{H} is called a decomposition into K -types.

8.4 Fourier Transforms and Convolutions

Now suppose that μ is a Borel probability measure on a locally compact group G , and ξ is an irreducible representation of G acting in \mathcal{H}_ξ . Then we may define the Fourier transform of μ at ξ by:

$$\hat{\mu}(\xi) = \int_G \xi(g^{-1})\mu(dg).$$

This is to be thought of as a Bochner integral and defines an operator in $\mathcal{L}(\mathcal{H}_\xi)$ by

$$\langle \hat{\mu}(\xi)v, w \rangle = \int_G \langle \xi(g^{-1})v, w \rangle \mu(dg),$$

for $v, w \in \mathcal{H}_\xi$.

In section 3 we use generalised Eisenstein integrals to express this operator in terms of its projections on the K -types.

If μ_1 and μ_2 are Borel probability measures on G , their convolution $\mu_1 * \mu_2$ is a Borel probability measure defined on G for which

$$\int_G f(g)(\mu_1 * \mu_2)(dg) = \int_G \int_G f(gh)\mu_1(dg)\mu_2(dh), \quad (8.42)$$

where f is an arbitrary bounded Borel measurable function on G .

It is now standard and easy to see that

$$(\widehat{\mu^{(1)} * \mu^{(2)}})(\xi) = \widehat{\mu^{(1)}}(\xi)\widehat{\mu^{(2)}}(\xi),$$

where the right hand side is composition of operators in $\mathcal{L}(\mathcal{H}_\xi)$.

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