Probability on Compact Lie Groups

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This book is dedicated to my family.

Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect.

Hermann Weyl, “Symmetry”.

You should meditate often on the connection of all things in the universe and their relationship to each other.

Marcus Aurelius, “Meditations”.

Preface *by Herbert Heyer*

Since the trendsetting monographs of Grenander, Parthasarathy and Heyer the study of probability measures on algebraic-topological structures such as topological semigroups, groups and hypergroups has developed into a comprehensive area of mathematical research, with applications ranging from the dynamics of stochastic processes to statistical decision theory. A basic tool in advancing the theory was the abstract harmonic analysis of locally compact groups, in particular the technique of Fourier transformation based on unitary representations. While the initial studies mostly dealt with general classes of groups and measures, the emphasis during the last four decades has been concretization in two directions: to choose special classes of groups and to consider special types of measures. In the seminal books of Diaconis, Hazod and Siebert, and Liao innovative contributions about random walks on, and Lévy processes in groups were included up to the actual state of the art. Originally there was a broad variety of problems of classical probability theory that called for generalization and standardization within the framework of locally compact groups. We mention selectively the embedding of infinitely divisible measure into convolution semigroups, the canonical decomposition of convolution semigroups, the central limit problem, and arithmetical characterizations of prominent types of probability measures. All of these problems could be more easily approached once the underlying group was Abelian, although some of the analysis had already been extended beyond this special case at an early stage. The author of the present book chooses compact Lie groups in order to describe recent advances of the theory. On the basis of current sources he discusses for example results on positive definiteness of mappings on duals of compact Lie groups, on regularity of densities of probability measures, and on deconvolution. Part survey and part treatise, the presentation supplements the existing expository literature on the subject. It invites the reader to learn first about compact Lie groups and their representations before it leads him to topics in structurally oriented probability theory. A wealth of references accompanying the exposition opens the eye of the reader towards future research. It is a pleasant experience to see well-established results and recent progress in the theory of probability measures on compact Lie groups summarized in book form. Of course in a handy publication the choice of relevant topics can only be subjective. Yet the author's very own research and his overall picture of the field guarantee successful reading for all who wish to absorb research in probability theory from an algebraic-topological point of view.

Contents

Introduction **7**

Introduction

Probability on groups gives a mathematical context to the interplay of *chance* with *symmetry*. To study this subject involves investigating probability measures on groups and random variables, stochastic processes and random fields that are group-valued. At the very least this entails the interaction of (probabilistic) measure theory and group theory, but the range of mathematical tools required to develop the subject is much wider than this and includes representation theory, harmonic analysis, functional analysis and differential geometry, as well as stochastic analysis and stochastic differential geometry. Furthermore the subject is highly applicable, and in the first two decades of the twenty-first century there has been considerable interest in applications to engineering, such as signal processing, robotics and the measurement of biological molecules. There have also been complementary developments in statistical estimation and inference on groups and manifolds which have been strongly motivated by these applications.

We can study probability theory in e.g. finite groups, algebraic groups, Lie groups, locally compact groups, loop groups and current groups, so the reader may well be asking – why *compact* Lie groups? There are a number of reasons for this. To begin with, this book is designed to be an *introduction*, and so it should avoid the temptations of generality. Indeed, the topic of probability theory on general locally compact groups is the theme of a classic monograph by Herbert Heyer [**?**], which although published in 1977 still remains highly relevant today. Compact Lie groups have the advantage that:

- They are the simplest class of continuous groups, beyond the abelian, to display the key feature of *noncommutativity* (in general) that leads to so much fascinating structure.

- They have a rather straightforward representation theory. In particular, their irreducible representations are all finite-dimensional. This leads to a *Fourier transform* (characteristic function) that is matrix-valued, while for more general groups this will operate in an infinite-dimensional space. Furthermore the theory of highest weights enables us to carry out standard analytic operations (such as taking limits) in the "Fourier co-variable".

- Every sequence (or net) of probability measures on a compact group is tight.

- There are many important and interesting examples. These include the tori Π^n, the special orthogonal groups $SO(n)$ and the special unitary groups $SU(n)$. The group $SO(3)$ of rotations in three dimensional space is of particular interest to engineers [**?**, **?**, **?**] and cosmologists [**?**]. The symmetry group of the standard model in elementary particle physics is the compact Lie group $U(1) \times SU(2) \times SU(3)$ [**?**].

Of course there is much that can be done with compact groups alone, but compact Lie groups admit a differential structure, and this enables us to exploit natural analytic/geometric structures such as the Laplacian and its associated heat semigroup and heat kernel. In fact, one can think of a Lie group informally as the natural context in which calculus meets symmetry, so one can, for example, describe symmetry transformations that preserve some invariant quantity, infinitesimally.

There are three key tools that are needed to study a probability measure μ or a process on a group G. The first is the Fourier transform as mentioned above, whose value at an irreducible representation π is $\widehat{\mu}(\pi) = \int_G \pi(g^{-1})\mu(dg)$. The second is functional analysis, which enters the picture when we consider convolution operators of the form $P_\mu f = f * \mu$ acting on various Banach spaces. Finally, we use stochastic analysis to study the interaction of systems with noise by means of stochastic differential, or stochastic partial differential equations. In this book we will almost entirely concentrate on the interplay between the first two of these tools and we will hardly use the third at all. Again there are some good reasons for this. Firstly this is an introductory book and I didn't want it to get too long. Secondly, from a pedagogic viewpoint, there is something to be said for imitating the process we go through as an undergraduate or first-year graduate student, where we take a course that covers topics like characteristic functions and the central limit theorem before we try to learn about stochastic integration. Thirdly, many topics that interest engineers and statisticians, such as deconvolution, seem to rely mainly on Fourier transform techniques. Finally, there was some intellectual curiosity on my own part – how far could I go without using one of my favourite tool kits? But the reader who gets to the end of the book will have encountered many places where stochastic analysis is needed to make further progress, and there are abundant references for further reading in this direction.

This book is specifically aimed at graduate students who have taken a standard course in measure-theoretic probability and have some training in functional analysis (particularly, Hilbert spaces) and a smattering of general topology, but do not necessarily have a strong background in group theory, representations and differential geometry. Of course it is also suitable for experts in Lie theory and associated harmonic analysis who want to find out what the probabilists are doing with their beautiful subject. Finally, I hope it will be accessible to the growing number of highly mathematically trained engineers, physicists and statisticians who are now working on applications of these ideas and who wish to fill in some gaps in their knowledge.

Guide To Reading The Book

Chapter 1 is a roller-coaster ride through the essentials of topological groups and Lie groups. Of course this is a huge topic and it cannot be learned in a hurry. I would encourage readers who are novices in this area to study an authoritative text as an accompaniment. Chapter 2 develops all the representation theory that we need for the book and proves one of the key results – the Peter-Weyl theorem. We also begin the important study of the Fourier transform of functions. Here I am highly indebted to Faraut's beautiful monograph [**?**], and large parts of the account are closely based on his approach. In the last part of the chapter, I introduce (in a somewhat condensed manner) the theory of weights and sketch the proofs of the wonderful character and dimension formulae of Hermann Weyl. Experts on Lie theory can skip these two chapters, but might want to glance at section 2.3 to remind themselves of key properties of the Fourier transform.

The short Chapter 3 introduces the important tool of the Laplacian and the associated Sobolev spaces. In the second part of the chapter we utilise the one-to-one correspondence between irreducible representations and dominant weights to regard the Fourier transform as a "function" of the latter. Since weights live in a finite-dimensional vector space, we can now carry out standard analytic operations, such as taking limits, on the "Fourier co-variable". In particular, we present Sugiura's [**?**] far-reaching characterisation of smooth functions on the group in terms of decay properties at infinity of the Fourier transform.

In Chapter 4, we at last turn our attention to probability measures on groups and make a detailed study of characteristic functions (i.e. Fourier transforms of measures). In particular, we describe a compact-group analogue of the famous Lévy convergence theorem that is due to Kawada and Itô [**?**]. A key theme of this chapter is absolute continuity, and we begin to see the importance of convolution operators through a beautiful theorem of Raikov [**?**] that tells us that the probability measure μ is absolutely continuous if and only if the associated convolution operator P_μ is compact in the space of continuous functions on the group. Using Sugiura's techniques, we then find necessary and sufficient conditions for a measure to have a smooth density in terms of the decay of the characteristic function. In the last part of the chapter we study random walks on general (not necessarily compact) Lie groups, and give some potential-theoretic characterisations of recurrence due to Guivarc'h et al. [**?**]

Standard Gaussian measures on compact Lie groups are naturally described through their Fourier transform in terms of eigenvalues of the Laplacian (the Casimir spectrum). They have smooth densities which can be easily obtained from the heat kernel. In recent years there has been increasing interest in both theoretical work and modelling with non-Gaussian processes, and Lévy processes (which are essentially stochastic processes with stationary and independent increments) have been at the heart of this activity. In Chapter 5, we study the analogues of these processes on general Lie groups but at the level of measures. So we investigate convolution semigroups of probability measures and the associated C_0-semigroups of convolution operators. We obtain Hunt's important characterisation [**?**] of the generators of these semigroups, which can be seen as an analogue of the classical Lévy-Khintchine formula on Euclidean space. As a corollary we derive a less well-known and more direct generalisation of the Lévy-Khintchine formula using the Fourier transform. We also develop Breuillard's [**?**] presentation of the central limit theorem, itself originally due to Wehn. Subordination is an important technique for obtaining new examples of interesting convolution semigroups, and specialists in mathematical finance know that the most interesting cases are obtained by subordinating Brownian motion. The same is true for Lie groups, and we examine some important classes (such as analogues of the stable laws) and establish smoothness of densities.

Finally Chapter 6 is a short introduction to statistics on compact Lie groups. It focusses on work by Kim and Richards [**?**] on deconvolution in the context of estimating the distribution of a signal from observations made on an output that is distorted by an independent noise.

Each chapter finishes with a brief guide to further reading, and there are many references in the bibliography for readers to delve further into the literature. I should perhaps apologise to anyone whose work I have neglected to mention. The bibliography, like the book that it references, is not meant to be an exhaustive archive of contributions to this very wide area. It is rather a guide to the available literature, and in many cases I have referenced particular papers and books in the knowledge that the reader can find many more interesting references in the bibliography of that paper or book. Despite my best efforts, this book will surely contain many typos and (hopefully minor) errors. Please send these to me at d.applebaum@sheffield,ac.uk. They will be posted on my website at http://www.applebaum.staff.shef.ac.uk/books.html

The book contains many proofs but in addition a large number of statements are included without proof, especially in Chapter 1. In particular I have tended to omit proofs that require a lot of structure theory, but include those that are more analytical, since I believe that this will make the volume more accessible for the typical reader, who is likely to be an analytically trained probabilist. Of course, where the proof is missing, there will be a reference to where it can be found in its full glory.

There are a number of appendices, and in particular, Appendices 1, 3, 5 and 6 give very heavily abridged accounts of those aspects of topology, differentiable manifolds, measures on locally compact spaces and compact operators in Hilbert space (respectively) that are needed. They can serve as a quick reminder of key facts, but are no substitute for systematic study of the topic in question. I have included references to standard texts for those who need them, but the reader can also find a fuller treatment of all of the above (and a great deal more, including a chapter on compact and locally compact groups, as well as representations of the former) in a two-volume work by Knapp [**?**, **?**].

I hope the reader will find that the book contains a joyful and stimulating selection of topics. However one should be aware that compact groups are, though important, a restrictive class, and many important applications involve groups such as the Heisenberg group or the Euclidean group (the semi-direct product of translations with rotations). Analogues of some results obtained in this book are not yet available in these contexts, and there is a great deal of work still to be done. In particular, it is important that mathematicians, statisticians and engineers do not work in isolation as there is so much that they can learn from each other.

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