Review of G.Kallianpur, P.Sundar, "Stochastic Analysis and Diffusion Processes," Oxford Graduate Texts in Mathematics **24**, Oxford University Press (2014); ISBN 978-0-965706-3 (hbk.), ISBN 978-0-965707-0 (pbk.); 352 pages.

Stochastic calculus was one of the great mathematical discoveries of the twentieth century. It enables random fluctuations to be systematically incorporated into ordinary and partial differential equations, and this opened up the way to a host of theoretical developments and practical applications, which are still being very actively explored. The book under review is a graduate text designed to present a self-contained course to beginners who already have some background in measure-theoretic probability and functional analysis. Sensibly, for the main part of the book, the authors deal only with Brownian motion as the basic noise process. This has the advantage of having a Gaussian distribution and (almost surely) continuous sample paths, as well as playing a starring role in many important applications, such as the celebrated Black-Scholes formula for option pricing. The first six chapters take a familiar path through the theory, but there are some welcome additions to the standard selection of topics. Brownian motion is introduced and constructed. This leads into a chapter on martingales. Then we switch from probability to analysis to meet an introduction to semigroups of operators, in order to study the relationship between Brownian motion, the heat kernel and the Laplace operator. Next we study the Itô stochastic integral, prove the celebrated Itô formula, and establish existence and uniqueness for strong solutions of stochastic differential equations (SDEs). The "welcome additions" that I mentioned above include a lovely derivation of the Kosmambi-Karhunen-Loéve expansion of Brownian motion, and a proof of the Yamada-Watanabe theorem which states that that existence of a weak solution of an SDE together with pathwise uniqueness implies existence of a unique strong solution.

In the last five chapters, the authors include some topics which are not typically found in books of this type, so I'll say a little more about these. Chapter 7 is about weak solutions of SDEs and the solution of the so-called martingale problem. This is one of the more technical areas of stochastic analysis and the authors have given a very accessible introduction to it. In chapter 8, the authors explore one of the most important areas of the subject, the relationship between SDEs and partial differential equations (PDEs). Roughly speaking, the SDE gives a precise description of the dynamics of a randomly evolving particle. The price we have to pay for this level of detail is *uncertainty*. Now if you average the solutions of the SDE over all possible paths, you obtain a PDE. You have sacrificed a detailed description of the dynamics, but gained a *deterministic* description. In practice, the two approaches complement each other. This is illustrated by the authors with an account of the Kolmogorov equations for a diffusion process, and the probabilistic solution of the classical Dirichlet problem.

In Chapter 9, I encountered a topic that I have never met before. Entitled "Gaussian solutions", the account begins by a discussion of some-well known functional analytic ideas - compact and Hilbert-Schmidt operators, Volterra integral operators, and the more specialised Goldberg-Krein factorisation. The main result is that a certain class of Gaussian solutions of SDEs have an explicit stochastic integral representation involving the kernsl of a Volterra integral operator.

Chapter 10 is the only one that departs from path-continuous processes, in order to give a swift introduction to processes with jumps. The authors introduce the Skorohod topology on the space of functions that are right continuous with left limits at every point. The key result here is the proof that any diffusion process can be weakly approximated by a sequence of jump processes. Invariant measures and ergodicity are important concepts in the study of dynamical systems and Chapter 11 focusses on these. An L^2-ergodic theorem is proved for one-dimensional diffusions. The authors also present a key theorem, due to Doob, that states that if the transition semigroup of a Markov process is both strongly Feller and irreducible, then there is a unique invariant measure (and hence the process is ergodic). The book concludes with a chapter on large deviations. This is the topic that investigates "good" rate functions that describe the exponential decay of probabilities of rare events. The authors explore the role played by relative entropy in providing a variational principle for a large class of Laplace functionals.

In their introduction, that authors remark that they wanted to write an "enjoyable" book, where this should also be taken to mean "readable and concise with full details". To a very large part I think they have succeeded. This is a very good book on which to base a graduate course, or to use for self-study. Most chapters conclude with about six to nine exercises. My only complaint is that sometimes the authors include results without proof (which is fine), but then they also omit a reference to where a proof might be found, e.g. the Dynkin class theorem in Chapter 1 and the Hille-Yosida theorem in Chapter 4. On the other hand, some important subsidiary results are proved in full within the text, such as Young's inequality in Chapter 8. The book concludes with a short section of notes on the chapters which directs the reader to some original sources that can be found in a fairly extensive bibliography. If I were giving a graduate course on this topic, then I would certainly use this book, but I would want to supplement it with some others to fill in a few gaps. For example, whether we like it or not, financial applications of this theory are popular with students who seek jobs in banking and investment. The book under review has only a one page summary of the Black-Scholes equation. The book by Steele [2] covers the same material as can be found in Chapters 1-6, but then gives a more in-depth account of the financial applications (including American options). Another topic that students should know about is simulation of paths, and they can find a very nice treatment of this in Chapter 20 of Schilling and Partzsch [1].

References

[1] R.L.Schilling, L.Partzsch, *Brownian Motion: An Introduction to Stochastic Processes*, W de Gruyter (2012)

[2] J. M. Steele, *Stochastic Calculus and Financial Applications*, Springer-Verlag (2001).