Corrections to "Lévy Processes and Stochastic Calculus" - by D.Applebaum *May 2005*

Most of these errors were found and corrected by Fangjun Xu, who is currently a first year graduate student at Nankai University. All page numbers refer to the first printing.

18,+7,-12 and 19, +2, +11 Change $0 \leqslant t_1 < t_2 < \cdots < t_n < \infty$ to $t_1, t_2, \ldots, t_n \in \mathbb{R}^+, t_1 \neq t_2 \neq \cdots \neq t_n$.

28, 14, Delete "$(\alpha(n), n \in \mathbb{N})$ be a sequence in $\mathbb{R}^d$" and replace with "U_n be a sequence of Borel sets in $\mathbb{R}^d$"

28, 16 and 17, Replace $[-\alpha(n), \alpha(n)]$ with U_n, in each case.

29, -7 Replace $b = 0$ with $b = c\int_{\hat{B}} x\mu(dx)$.

31,-6 Insert i.i.d. between "real-valued" and "random".

35, 13 Note that this is the correct formula only in the rotationally invariant case. For the general case see Sato, p. 78 - equation (14.4).

35,-2 Change $\tan\left(\frac{\pi\alpha}{2}\right)]\,\mathrm{sgn}(u, s)$ to $\tan\left(\frac{\pi\alpha}{2}\right)\mathrm{sgn}(u, s)]$

37, +7 In fact $|u| \leqslant 1$ can be replaced by $0 \leqslant u \leqslant 1$ (still due to Riemann).

37, +10 Delete "Chapter 9 of" (the ghost of a departed reference !)

38-39 The argument used to prove that the mapping $E_{m,c}$ is conditionally positive definite is incorrect (the last inequality on line 5 of page 39 is wrong). A correct derivation will appear in the second edition. For now observe that the result also follows from the argument given later in Example 1.3.29 on page 56.

52, 7 The right hand side should be

$$\exp\left[-t\int_0^\infty (1 - e^{-ux})ax^{-1}e^{-bx}dx\right].$$

55, -2 In the first term, replace $-s(-\eta_X(u))$ with $s\eta_X(u)$, and in the second term replace $\eta_X(u)$ with $(-\eta_X(u))$.

56, Delete lines +1 and +2.

65,-1 Replace X with M.

73,+1 Replace $\eta(u)$ with $\mathrm{Re}\,(\eta(u))$.

74, +12 and +13 Of course $q^2 = 4$ here.

78,+2 Replace $X(t)$ with $M(t)$.

80,-10 Replace $\leqslant \infty$ with $< \infty$.

81, -9 The square should be outside, and not inside, the expectation.

82,-2 Replace 2.1.4(2) with 2.1.4(3).

85, +12 Insert "integer-valued" before Lévy process.

85,+15 Replace $N(t + T_{n-1})$ with $N(t)$.

88,+11 Change $= n$ to $\geqslant n$.

89, -3 I've had quite a lot of correspondence about this. For the record, M is not a measure - the set of measure zero on which this fails depends on the choice of $(A_n, n \in \mathbb{N})$ and the union of all of these sets may well be the whole space !
However there is no harm in replacing the a.s. sigma-additivity here with finite additivity, as this is all we require for applications later on.

90, -19 to -18. After "algebra.", insert " Let V be a fixed element of $\mathcal{C}$ and". Also delete following "Let" and replace by "let". Replace $\mathbb{R}^+ \times U$ with $\mathbb{R}^+ \times (U - V)$.

90, -17 Delete "there exists $V \in \mathcal{C}$ such that".

90, -13 Delete "$-\{0\}$".

91,-4 Replace rhs with

$$\exp\left[t\int_{\mathbb{R}^d}(e^{i(u,x)}-1)\mu_{f,A}(dx)\right],$$

91,-3 Delete and replace with
"where $\mu_{f,A}(B) = \mu(A \cap f^{-1}(B))$, for each $B \in \mathcal{B}(\mathbb{R}^d)$."

94,-6 On rhs of (2.9), replace $\int_A$ with $\int_{\mathbb{R}^d}$ and μ_f with $\mu_{f,A}$.

96, -6 After "martingales" insert ", where each $M_j(0) = 0$(a.s.)"

99,+3 Replace $t\int_A f(x)\nu(dx)$ with $t\left|\int_A f(x)\nu(dx)\right|$.

101, -12 Before "We note..." insert "We first assume that $T_1 < \infty$ (a.s.)"

102, -5 After ∞ but before $\square$, insert the following:
If it is not the case that $T_1 < \infty$ (a.s.), we first argue as above on the event $T_1 < \infty$ and use

$$\mathbb{E}(|X(t)|^m \chi_{\{T_1=\infty\}}) \leqslant C^m P(T_1 = \infty) \leqslant C^m,$$

for all $t \geqslant 0$, hence

$$\mathbb{E}(|X(t)|^m) = \mathbb{E}(|X(t)|^m \chi_{\{T_1<\infty\}}) + \mathbb{E}(|X(t)|^m \chi_{\{T_1=\infty\}}) < \infty.$$

102,-6 Replace $\int_{2rC}^{2(r+1)C}$ with $\int_{2rC \leqslant |x| < 2(r+1)C}$.

104, +7 Change $\epsilon_1 < 1$ to $\epsilon_1 = 1$.

104, + 8 Change $\leqslant \epsilon_m$ to $< \epsilon_m$.

106,+2 Insert t in front of the integral on the rhs.

110,-8 Replace "$b = gb$" with "$b = gb + \int_{\mathbb{R}^d-\{0\}}[gy(\chi_B(gy) - \chi_B(y)]\nu(dy)$". After invariant., insert "In the case where G acts as a group of isometries, the first of these conditions reduces to $b = gb$."

118. The proof of theorem 2.8.1 isn't quite right. A much better proof (to be included in the second edition) can be found in Billingsley [44], p.110.

122,+2 Change $|g||$ to $||g||$.

122,-12 Change $p_{s,t}(A, x)$ to $p_{s,t}(x, A)$.

124, +130 t_0 is of course 0 (no loss of generality here).

125, Delete the eight and a bit lines following "Markov." and up to (but not including) "and the result...".Replace as follows:

Let $0 \leqslant s < t < \infty$ and let $\mathcal{R}_s$ be the collection of all cylinder sets of the form $A = I_{t_0,t_1,\ldots,t_n}^{A_1,A_2,\ldots,A_n}$ with $0 = t_0 < t_1 < \ldots t_n < s$. For each $f \in B_b(\mathbb{R}^d)$, $A \in \mathcal{R}_s$ we have,

$$\begin{aligned}
& \mathbb{E}(\chi_A \mathbb{E}(f(X(t))|\mathcal{F}_s) \\
= \ & \mathbb{E}(\chi_A f(X(t))) \\
= \ & \mathbb{E}(\chi_{\{X(t_0)\in A_0, X(t_1)\in A_1,\ldots,X(t_n)\in A_n\}} f(X(t))) \\
= \ & \int_{A_0} \mu(dx_0) \int_{A_1} p_{t_0,t_1}(x_0, dx_1) \ldots \int_{A_n} p_{t_{n-1},t_n}(x_{n-1}, dx_n) \int_{\mathbb{R}^d} f(y) p_{t_n,t}(x_n, dy)
\end{aligned}$$

On the other hand

$$\begin{aligned}
& \mathbb{E}(\chi_A \mathbb{E}(f(X(t))|X(s)) \\
= \ & \mathbb{E}(\chi_{\{X(t_0)\in A_0, X(t_1)\in A_1,\ldots,X(t_n)\in A_n\}} \mathbb{E}(f(X(t))|X(s))) \\
= \ & \int_{A_0} \mu(dx_0) \int_{A_1} p_{t_0,t_1}(x_0, dx_1) \ldots \int_{A_n} p_{t_{n-1},t_n}(x_{n-1}, dx_n) \int_{\mathbb{R}^d} p_{t_n,s}(x_n, dx) \\
& \int_{\mathbb{R}^d} f(y) p_{s,t}(x, dy) \\
= \ & \int_{A_0} \mu(dx_0) \int_{A_1} p_{t_0,t_1}(x_0, dx_1) \ldots \int_{A_n} p_{t_{n-1},t_n}(x_{n-1}, dx_n) \int_{\mathbb{R}^d} f(y) p_{t_n,t}(x_n, dy),
\end{aligned}$$

by Fubini's theorem and the Chapman-Kolmogorov equations. So we have deduced that

$$\mathbb{E}(\chi_A \mathbb{E}(f(X(t))|\mathcal{F}_s) = \mathbb{E}(\chi_A \mathbb{E}(f(X(t))|X(s)),$$

for all $A \in \mathcal{R}_s$. Now $\mathcal{R}_s$ forms a so-called π-system (i.e. it is stable under intersections). Since it generates the σ-algebra $\mathcal{F}_s$

we may appeal to a theorem of Dynkin's (see e.g lemma 1.8 in Rogers and Williams [261]) and use monotone convergence to conclude that

$$\mathbb{E}(\chi_A \mathbb{E}(f(X(t))|\mathcal{F}_s) = \mathbb{E}(\chi_A \mathbb{E}(f(X(t))|X(s)),$$

for all $A \in \mathcal{F}_s$,

128,+14 Change $\widetilde{p_{s,t}}(\{\Delta\}, \mathbb{R}^d)$ to $\widetilde{p_{s,t}}(\Delta, \mathbb{R}^d)$

130, +8 Change $M > 0$ to $M > 1$.

132,-1 On the rhs, change T_h to T_t.

134, 6 Replace $\overline{D_A}$ with $\overline{D}$.

134,-11, Replace G_A with D_A.

138,+6 Insert $\int_{\mathbb{R}^d}$ before $f(y)$.

147,-9 On rhs, replace $(T_s^X f)(s)$ with $(T_s^X f)(x)$.

149-5 to 150 +4. Delete and replace as follows

$$\begin{aligned}
&\leqslant \int_B \left(\int_{\mathbb{R}^d} |f(x+y) - f(x)|^p dx \right) q_t(dy) \\
&+ \int_{B^c} \left(\int_{\mathbb{R}^d} |f(x+y) - f(x)|^p dx \right) q_t(dy) \\
&\leqslant \int_B \left(\int_{\mathbb{R}^d} |f(x+y) - f(x)|^p dx \right) q_t(dy) \\
&+ \int_{B^c} \left(\int_{\mathbb{R}^d} 2^p \max\{|f(x+y)|^p, |f(x)|^p\} dx \right) q_t(dy) \\
&\leqslant \sup_{y \in B} \int_{\mathbb{R}^d} |f(x+y) - f(x)|^p dx + 2^p ||f||_p^p q_t(B^c),
\end{aligned}$$

By choosing B to have sufficiently small radius, we obtain $\lim_{t \downarrow 0} ||T_t f - f||_p = 0$ from the continuity of f and dominated convergence in the first term and the weak continuity of $(q_t, t \geqslant 0)$ in the second term, just as in the proof of Theorem 3.1.9.

166. +2 Replace $\int_{\mathbb{R}^d - \{0\}} [f(x+y) - f(x)] \nu(dy)$ with

$\frac{1}{2}\int_{\mathbb{R}^d-\{0\}}[f(x+y)-f(x-y)-2f(x)]\nu(dy)$. Make a similar change on line -10.

166, +8 Replace $(\mathbb{R}^d \times \mathbb{R}^d) - D$ with $\mathbb{R}^d \times (\mathbb{R}^d - \{0\})$.

175 Change $1 + \mathrm{Re}\,(\eta(u))$ to $1 - \mathrm{Re}\,(\eta(u))$ on lines 7, 12, 15 and 17.

196, +14 Change $||f_n - f||_2$ to $||f_n - f||_2^2$.

199, +3 Change $\sum_{j,k=1}^{n}$ to $\sum_{j,k=1}^{m,n}$.

199, +4 Change $\sum_{j}^{n}$ to $\sum_{j,k=1}^{m,n}$.

203,+5 Change $\int_{0,T}$ to $\int_0^T$.

204,+5 Change $F(t,x)$ and $\rho(dt,dx)$ to $F(s,x)$ and $\rho(ds,dx)$.

208,-6 Insert 4 before $\mathbb{E}$.

208,-5 Insert 4 after $=$.

210,+4 Replace $\sum_{n=0}^{m(n)}$ with $\sum_{j=0}^{m(n)}$.

224,+5 Change $f(t \wedge W(T_n^A-))$ to $f(W(t \wedge T_n^A-))$.

226,+12 Change (t,x) to (s,x).

228,-6 Change $\sum_{0\leqslant s\leqslant t} K^i$ to K^i.

229, -3 Delete this line.

234, +4 Change $M(t)$ to $M(t) - M(s)$ on left hand side.

237,+9 Insert $\int_0^t$ after $=$.

240,-14 Change $\mathbb{E}(M(t)|\mathcal{F}^s) = M(s)$ to $\mathbb{E}(M(s)|\mathcal{F}^t) = M(t)$.

245, +2, Delete "it clearly coincides with" and replace with "be aware that it may not necessarily coincide with".

248,+5 Insert K in front of $\sum_{0\leqslant s\leqslant t}$.

249,+6 Insert] after dt.

249, -9 Delete] after dt.

251, +9 Change (LM1) to (LM2).

252,-3 Insert $+$ after $G(s)$.

257,-13 Change $-$ to $+$.

260,-5 Change "to h" to "to P".

263,-2 Delete the last).

264,+14 Change $e^{u_n u_{n-1} t}$ to $e^{u_n u_{n-1} t_{n-1}}$.

273,+2 Change (SE1) to (SE)

273, +5 Change $x^2 \vee x$ to $x^2 \vee |x|$. In fact, it is sufficient to impose the condition $\int_{(c,-1]\cup[1,\infty)} x^2 \nu(dx) < \infty$.

273,+11, Change $[c,-1]$ to $(c,-1]$.

275, +8 Change both instances of $\tilde{S}(t)$ on rhs to $\tilde{S}(t-)$.

278,-3 Change $N(-\sigma^2 T/2, \sigma^2 T/2)$ to $N(-\sigma^2 T/2, \sigma^2 T)$

278,-1 Change $\log(k/s)$ to $\log(k/s) - rt$.

285,-10 In formula for $V(0)$, insert s in front of the first integral on the right hand side.

304, -11 Change $|x_2|$ to $|x_2|^2$.

304,-7 Change $=$ to $\leqslant$.

311, -15 Before "There exists", insert "If $c > 0$"

327,-10 Change $\Phi_{0,t+s}(y, \theta_{t+s}(\omega), \theta_{t+s}(\omega))$ to $(\Phi_{0,t+s}(y, \theta_{t+s}(\omega)), \theta_{t+s}(\omega))$

327,-9 Insert (after $=$.

330, -7, Delete | after $\mathbb{E}($.

330. Change dx to $\nu(dx)$ on lines -2, -6 and -7.

331. Change dx to $\nu(dx)$ on lines +2 and +3.

332, -10, Change $\Psi_{s,t}$ to $\Psi_{s,u}$ twice on left hand side.

335 Change M^d to M^D on lines 6 and 9.

344, +11 Change Y^c to Y_c.

344, -2, Change $Y^c s + t)$ to $Y_c(s + t)$.

345, -9 Change $f(\Phi_{0,t})$ to $M_f(t)$ and change $d(\Phi_{0,t})$ to $df(\Phi_{0,t})$ and insert $-$ in front of $\int_0^t$.

351,-7 Change $F(ux, y_1) - F(ux, y_2)$ to $F(y_1, ux) - F(y_2, ux)$.