Corrections/Comments re. "Lévy Processes and Stochastic Calculus (second edition)" - by D.Applebaum *August 2009*

(1) *Errata Included in New Printing to Appear in 2011*

p.43, +4 $< t_2 \leqslant \ldots$ should read $< t_2 < \ldots$.

p.49. Example 1.3.10. Of course $Y(t) = 0$ when $N(t) = 0$.

p.99, -5 This is a little misleading as on the next page we show that $\Delta X(t) = 0$ (a.s.) so it is (trivially) a Lévy process. The point of Exercise 2.3.1 is to prepare the way for that as the breakdown of independent increments follows only if you assume that $P(\Delta N(t_1) = 1) > 0$.

p.104, -13 σ-finite should read σ-additive.

p.113,-6 Such a sequence of partitions cannot exist so the argument given here is incorrect for establishing the result on A^c. Instead argue as on p.114 where the points in A are replaced by those in $A^c \cap \mathbb{Q}$. In fact, this argument is now easier as each $\Delta M_i(t_n) = 0$ when $t_n \in A^c \cap \mathbb{Q}$.

p.133, +7 $< \infty$ is missing after $\int_{|x|>1} |x|\nu(dx)$.

p.140, -8 Item (4). Its true that every càdlàg function on a finite closed interval is bounded but it is false that bounds are always attained. For a counter-example, consider f defined on $[0, 2]$ by

$$f(x) = x\chi_{[0,1)}(x) + \frac{1}{2}\chi_{[1,2)}(x).$$

Then $\sup_{x\in[0,2]} f(x) = 1$ and this is clearly not attained. This error doesn't have any effect on the rest of the book.

p.206, -11 T should be *densely defined.*

p.218, +6 It would be better notation to replace $F(t_j)$ on the right hand side with F_j. The reason for this can be seen more easily on line +11 and observing that if you take $t = t_j$ then $F(t_j) = F_{t_{j-1}}$.

p.228, -5 For a martingale-valued measure M to be *continuous* means that the process $(M_A(t), t \geqslant 0)$ is continuous for each $A \in \mathcal{A}$ (see p.105 for notation.)

p.230, +5 Replace $G(t) \in L^1[0,T]$ with $t \to G(t)$ is almost surely integrable on $[0,T]$.

p.255, In lines 6-7 all instances of $\Delta Y(s)$ should be replaced by $\Delta L(s)$ where $L(s) = \int_{|x|\leqslant 1} x\tilde{N}(s,dx) + \int_{|x|>1} xN(s,dx)$.

p.364 The mappings $F^i, G^i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ can be replaced by $F^i, G^i : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ to allow a Poisson random measure that lives on $\mathbb{R}^+ \times (\mathbb{R}^m - \{0\})$ and the theory goes through just as in the book. This is used implicitly in the section on "SDEs driven by Lévy processes" on pp.377-8.

p.405, +12 It is better to replace $C_0(\mathbb{R}^d)$ with the set of non-negative functions in $B_b(\mathbb{R}^d)$ with the understanding that both sides of the equation on line 11 may be infinite when μ is σ-finite.

(2) *More Recent Errata Discovered Too Late for New Printing*

All those in Chapters 1, 2 and 3 were sent in by Adam Majewski from Gdansk.

21, -6 There should be an additional $\int_{\mathbb{R}^d}$ on the right hand side of (1.4).

21, -3 The final μ_2 in (1.5) should be μ_1.

30, +7 Replace "Borel" with "open". Note that monotonic decreasing to $\{0\}$ means $U(n+1) \subseteq U(n)$ and $\bigcap_{n\in\mathbb{N}} U(n) = \{0\}$. So $0 \in U(n)$ for all n and since each $U(n)$ is open it contains an open ball of radius r_n centered on 0. It follows that each integral over U_n^c is finite in the next display.

31, +8 Replace χ_B with $\chi_{\widehat{B}}$.

44, -6 Replace "and that, for each $u \in \mathbb{R}^d, t \geqslant 0$. Define" with "and for each $u \in \mathbb{R}^d, t \geqslant 0$ define".

47, -1 Include "for each $t \in [0,1]$".

50, -7. Replace $T_m^{(1)} = T_m^{(2)}$ (a.s) with $P(T_m^{(1)} = T_m^{(2)}) > 0$.

50, -6 Replace $Z(m+n-2) = 2$ (a.s.) with $P(Z(m+n-2) = 2) > 0$.

58, +6 Replace $a \in \mathbb{R}^d$ with $a > 0$.

58, -3 Replace $\eta_{Z(t)(u)}$ with $(u, Z(t))$.

66, +14 Replace $X\left(\frac{1}{n}\right) + X\left(\frac{1}{n}\right) + \cdots + X\left(\frac{1}{n}\right)$ with $X\left(\frac{1}{n}\right) + \left(X\left(\frac{2}{n}\right) - X\left(\frac{1}{n}\right)\right) + \cdots + \left(X(1) - \left(X(\frac{n-1}{n}\right)\right)$.

66, +16 Replace $X\left(\frac{1}{n}\right) + X\left(\frac{1}{n}\right) + \cdots + X\left(\frac{1}{n}\right)$ with $X\left(\frac{1}{n}\right) + \left(X\left(\frac{2}{n}\right) - X\left(\frac{1}{n}\right)\right) + \cdots + \left(X\left(\frac{r}{n}\right) - \left(X(\frac{r-1}{n}\right)\right)$.

66, +17 Insert a.s. after $X(q) > 0$.

94, -13 Replace $\mathbb{E}(\langle M, N\rangle(t)^2)$ with $\mathbb{E}(\langle M, N\rangle(t))^2$

99, + 9 We don't really use induction here so replace "Now...that" with "We now show that for all $n \in \mathbb{N}$"

102, + 7 Change $X(w) - X(u)$ to $X(u) - X(w)$ in (2.4).

102, +9 Change $n-1$ to $n+1$.

104, -6 Change $\lambda(B)$ to $\lambda(A)$.

104, -3 Change "sets in S" to "sets in $\mathcal{S}$".

105, +3 Change $\lambda(dx, dt)$ to $\lambda(dt, dx)$.

111, -10 Replace the sentence "A process...analogously." with "A process is of *infinite variation* if it fails to be of finite variation."

114, -8 Change $(M_1$ to M_1.

115, +2 and +3 Change $V_{M_2}(t))$ to $V_{M_2}(t)$.

119, +5 This uses a slight variation on the Chebyshev-Markov inequality to the effect that for a non-negative random variable $Y, P(Y < y) \leqslant e^y \mathbb{E}(e^{-yY})$. It is proved in the same way.

122, -10 Change $\int_{|x|<1} \tilde{N}(t, dx)$ to $\int_{|x|<1} x\tilde{N}(t, dx)$.

125, -9 Change j to k.

127, +11 $e^{it(u,b)}$ is missing from the right hand side.

132, +5 Change $\mathbb{E}(|Y^n|)$ to $\mathbb{E}(|Y|^n)$.

134, -1 Change x^2 to $|x|^2$ within the integral.

135, +9 Change $\Delta Y(t)$ to $\Delta Y_n(t)$.

135, -9 Change T_2 to T_n^2.

135, -1 and 136, +3 Change $\epsilon_{n+1} < |x| < \epsilon_n$ to $\epsilon_{n+1} \leqslant |x| < \epsilon_n$.

137, +11 Of course M must be a square-integrable *martingale* here.

140, -11 Delete the second sentence of (2), e.g. $g \in \mathcal{D}(0, 2)$ where $g(x) = (1 - x)\chi_{[0,1)}(x) + \chi_{[1,2]}(x)$ but $\frac{1}{g}$ has infinite left limit at $x = 1$.

154, -12 Change $\mathbb{R}^d$ to $\mathbb{R}$.

160, -1 Insert "definite" after "positive" and change C to $\mathbb{C}$.

169, +4 Change $\mathbf{C}$ to $\mathbb{C}$.

194, -8 A dy is missing at the end of the RHS of (3.30).

203, +1 Change "T is bounded" to "$\tilde{T}$ is bounded".

308, +3-+4 $P = P^2 = P$ should read $P = P^2 = P^*$.

314, -9 $\mathcal{F}$ should be $\mathcal{F}_T$.

384, + 13 "Example6.4.1" should read "Example 6.4.1."

401, -8 $T_{0,t}$ and $S_{0,t-s}$ should be T_t and S_{t-s} (respectively).

(3) *Errata Found in the New Printing (2011)*

Most of these were discovered by Christian Fonseca Mora. They should all be put right in a new printing to appear in 2013.

p.4, -8 (ii) should be (iii)

p.6, +6 Some text has been omitted. After $\mathbb{R}$ it should read "It becomes a normed space (in fact a Banach space) with respect to..."

p.6, -4 The integral should only be defined at this stage for non-negative simple functions, i.e. those for which each component of the vector c_j is non-negative. $(1 \leqslant j \leqslant n)$.

p.88, +8 $X(t)(\omega)$ should be $X(s)(\omega)$.

p.114 Unfortunately A^c has been mistyped as A^0 on lines +6 and +11.

p.114, -2, -3 (twice) and -5, change $V_{M_2}(t))$ to $V_{M_2}(t)$.

p.129, +9 "Exercise 2.3.15" should read "Example 2.3.15."

p.133, +11 Change $\mathbb{E}(|X(t)|^2$ to $\mathbb{E}(|X(t)|^2)$.

p.144, -7 Change $t \geqslant 0$ to $0 \leqslant s \leqslant t < \infty$.

p.144, -6 Change "(1), (2), (3) and (4)" to "(1), (2), (4) and (6)".

p.148, +13 Change $\mathcal{F}_s)$ to $\mathcal{F}_s))$.

p.171, +14 Change T_s^X to T_s.

p.173, -11 Change $(T_t \geqslant 0)$ to $(T_t, t \geqslant 0)$.

p.177, -5 Change $g \leqslant 0$ to $g \geqslant 0$.

p.178, -6 Change $X(t))$ to $X(t)))$.

p.187, +7 and +11, Change uy to (u, y).

p.206, +13 Change $\psi \in B$ to $\phi \in B$.

p.208, line 1 Change f to ϕ and g to ψ.

p.222 lines 3 and 4. Change $F(t_l)$ to $F_p(t_l)$.

p.222 lines 4 and 8. The second instance of A_k should be A_p.

p.244 line 12 and p.245 line 10, Change $\sum_{j=0}^{n}$ to $\sum_{j=0}^{m(n)}$.

p.246, -13. Delete , between d and $\}$.

p.249, -4 Delete the extra) after the first $f(W(t \wedge T_n^A-)$.

p.254, -4 Delete) just before ds.

p.269, +10. Insert (after {.

p.269, -3 Replace $\beta N(s, A)$ by $\beta N(s, B)$.

p.281, +7 Replace E_Y by $\mathcal{E}_Y$ (both instances).

p.288 The equation label (5.5) should be deleted and (5.6) should be (5.5).

p.304, +5 Change $\psi_n^{(1)}, n \in \mathbb{N}$ to $(\psi_n^{(1)}, n \in \mathbb{N})$.

p.309, -6 Change Appendix 5.7 to Appendix 5.9.

p.315, +5 Change t_1 and t_{n+1} to s_1 and s_{n+1} (respectively).

p.316, +9 Insert). after $|^2$.

p.317, +4 Delete ' before Lévy.

p.330, -11 Change $\log(S(t)$ to $\log(S(t))$.

p.330, -10 Change log[to log(.

p.352, +9 Change Yu to Yang.

p.358, -5 Change $\mathbb{R}$ ro $\mathbb{R}^d$.

p.372, -8 Replace n with N.

p.378, -8 Insert) before dt.

p.381, -4 Insert) before ds.

p.383, +7 Replace int with $\int$.

p.397, +7 and +13, Delete (just after sup.

p.397, +7 Insert) just after $b(\Psi_{s,u-}(y)$.

p.407, -4 Replace $T_{s,s+t}$ with $T_{s,s+t}^c$.

p.411, -2 Replace H_i with F_i.

p.414, +7 Replace X with 6.8.1.

p.415, -6 Replace σ with Σ.

p.417, -7 Replace $dX_d^j(s)$ with $dX_d^j(t)$.

p.420, +1 [346] should be [345].

p.421, +4 Replace y_2 with y_2).

p.421, -10 Insert P_2 after $\leqslant$.

p.422, -11 and -12, Change $\xi^i(x)$ to $\xi(x)$.

p.423, +3 [23] should be [22].

p.440. Item [194] Change Zurek to Jurek.

(4) *Errata Found To Late for Inclusion in 2013 Printing.*

xvi, +6 There should be no dt on the right hand side of the ode.

p.2, -4. The definition of open set in the relative topology is incorrect. It should read that U is open in S if $U = V \cap S$ where V is open in $\mathbb{R}^d$.

p.9, -7 The measure μ should be σ-finite for this sufficient condition to hold. See e.g. Prop. 3.4.5. in Cohn (p.110).

p.12, Theorem 1.1.7. For Fubini's theorem to hold, the measures μ_i should be σ-finite. Similarly μ should be σ-finite in Theorem 1.1.8 on page 13.

p.17, In Lemma 1.1.11 (1), $u_1, \ldots, u_d \in \mathbb{R}^d$ should be replaced by $u_1, \ldots, u_n \in \mathbb{R}^d, n \in \mathbb{N}$.

p.29. The wording at the beginning of Theorem 1.2.14 is misleading. It would be better to phrase it "If $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is infinitely divisible, then there exists..."

p.61, -7 $\int_0^\infty (|y| \wedge 1) m_{X,T}(dy) < \infty$ should read $\int_{\mathbb{R}^d - \{0\}} (|y| \wedge 1) m_{X,T}(dy) < \infty$. In fact this stronger condition cannot be derived from the condition given below. It only seems to hold under the assumption

$$\int_0^\infty (t^{\frac{1}{2}} \wedge 1)\lambda(dt) < \infty ...(*)$$

Instead of the stated condition from Sato, we need

$$\mathbb{E}(|X(t)|; |X(t)| \leqslant 1) \leqslant C(t^{\frac{1}{2}} \wedge 1),$$

which is (30.13) on p.198 therein. We then find that

$$\int_{\mathbb{R}^d-\{0\}} (|y| \wedge 1) m_{X,T}(dy) = \int_0^\infty \mathbb{E}(|X(t)| \wedge 1)\lambda(dt) \leqslant \int_0^\infty (t^{\frac{1}{2}} \wedge 1)\lambda(dt) < \infty.$$

Condition (*) is then also required for Theorem 1.3.33, but I conjecture that this result still holds in full generality. Can anyone see how to prove that? This is not needed anywhere else in the book. [For example, if $(T(t), t \geqslant 0)$ is an α-stable subordinator, then (i) imposes the requirement $\alpha \leqslant 1/2$.]

p.67, +11 The event on the rhs should be enclosed within $P(\cdots)$.

p.105, -4 Replace "σ-finite" with "σ-additive."

p.107, +2 Delete $|\cdot|$ inside Var. It should read $\mathrm{Var}\big(\int_A f(x)N(t,dx)\big)$.

p.124. The use of Taylor series here is incorrect. Instead, we should use the identity (for all $y \in \mathbb{R}, m \in \mathbb{N}$)

$$e^{iy} = \sum_{k=0}^{m-1} \frac{(iy)^k}{k!} + \theta \frac{|y|^m}{m!},$$

where $\theta \in \mathbb{C}$ with $|\theta| \leqslant 1$. This is proved in Sato p.40.
Then $I_1(t)$ and $I_2(t)$ are as before but $I_3(t)$ should be
$\theta \sum_{j=0}^{n-1} e^{iuY_c(t_j)}[\Delta Y_c(t_j)]^3$.
The analysis of $I_3(t)$ on p.125 is not much different to the published account.

p.140, +10 In Theorem 2.9.2 (i) replace $\Delta f(t)$ with $|\Delta f(t)|$; note also that $\Delta f(t) > k \Rightarrow |\Delta f(t)| > k$, which can be seen by replacing f with $-f$ in the former.

p.145, +11 We should here take a regular version of the conditional probability - see Kallenberg pp.106-7.

pp.182–3 In the statement of Theorems 3.5.3 and 3.5.4, the space $C_0(\mathbb{R}^d)$ should be replaced by $C(\mathbb{R}^d)$. The conditions on μ in (3.20) and η in (3.22) are not strong enough to ensure that $Af \in C_0(\mathbb{R}^d)$.

p.207, -4 The measures $\mu_{\phi,\psi}$ are *signed.* In fact from the polarisation identity, we get the Jordan decomposition

$$\mu_{\phi,\psi}(\cdot) = \frac{1}{4}||P(\cdot)(\psi + \phi)||^2 - \frac{1}{4}||P(\cdot)(\psi - \phi)||^2.$$

p.212, +2 Replace $\exp(-x^2)$ with $\exp(-|x|^2)$.

p.247, + 1 "adapted" should read "measurable".

p.270, -9 The t at the end of the line should be s.

p.281 - 8 The right hand side should be written $\exp(C(t) + D(t))$, where

$$C(t) = \sum_{0 \leqslant s \leqslant t} \{\log\left[1 + \Delta Y(s)\right] - \Delta Y(s)\}\chi_{\{|\Delta Y(s)| \geqslant 1/2\}},$$

$$D(t) = \sum_{0 \leqslant s \leqslant t} \{\log\left[1 + \Delta Y(s)\right] - \Delta Y(s)\}\chi_{\{|\Delta Y(s)| < 1/2\}}.$$

The proof goes through pretty much unchanged, with minor adjustments.

p.374, -9 The formula $Y(t) = Y(\tau_1) = Z_1(t) - Z_1(\tau_1)$ for $\tau_1 < t < \tau_2$ is incorrect. In fact we don't need the process Z_1 at all. Instead we should use the stochastic flow $(\Psi_{s,t}; 0 \leqslant s \leqslant t < \infty)$ corresponding to the process $(Z(t), t \geqslant 0)$. This is formed exactly as in (6.32), but with no N term. Then the correct formula for the process is:

$$Y(t) = \Psi_{\tau_1,t}(Y_{\tau_1}), \text{ for } \tau_1 < t < \tau_2,$$

and a similar formula holds between later stopping times. My thanks to Tomasz Kosmala of Kings College, London for spotting the error.

p.385, -0 to -11, In the display that starts two lines below (6.32), all four integrals on the right hand side should be from s to t, and not from 0 to t.

p.408. The argument at the end of the proof of Theorem 6.7.9 to show that $\mathcal{L}_c := \mathcal{L} + c$ is the generator of $(T_t^c, t \geqslant 0)$ is not correct. Here is a sketch of a more convincing argument. First note that by standard perturbation theory arguments (see e.g. Theorem 3.1 in Davies [85], pp.68–9), since c is bounded $\mathcal{L}_c$ generates a C_0–semigroup. Hence there is a unique solution to the equation

$$\frac{\partial u(t)}{\partial t} = \mathcal{L}_c u(t), \quad \text{with} \quad u(0) = f.$$

Now follow the argument in Durrett [99] p.138–9. For fixed $t \geqslant 0$ and $0 \leqslant s \leqslant t$, define the process

$$M(s) = \exp\left\{-\int_0^s c(\Phi_{0,u}(y))du\right\} u(t-s, \Phi_s(y)).$$

By Itô's formula:

$$dM(s) = -c(\Phi_{0,s}(y))M(s)ds - \partial_s u(t-s, \Phi_s(y))ds + \mathcal{L}u(t-s, \Phi_s(y))ds + dN(s),$$

where $N = (N(s), 0 \leqslant s \leqslant t)$ is a martingale. The ds terms cancel, hence $(M(s), 0 \leqslant s \leqslant t)$ is a martingale, and we have

$$\begin{aligned} u(t,y) &= M(0) \\ &= \mathbb{E}(M(t)) \\ &= \mathbb{E}\left(\exp\left\{-\int_0^t c(\Phi_{0,u}(y))du\right\} u(0, \Phi_{0,t}(y))\right) \\ &= \mathbb{E}\left(\exp\left\{-\int_0^t c(\Phi_{0,u}(y))du\right\} f(\Phi_{0,t}(y))\right) \end{aligned}$$

from which it follows that $u(t) = T_t^c f$ and the result follows.

p.417, -6. In the final sum of (6.44), both of the first two terms are missing a superscript i.

p,422, -13. Put d in front of $f(\Phi_{s,t}(y))$ on LHS of display and delete $f(y)$ on RHS. Alternatively, insert $\int_s^t$ in front of last four terms on RHS.

p.422, -11. Change N to $\tilde{N}$ (so the two integrals on the right hand side of the display, may in fact, be combined into one.)