# Probability Measures on Compact Groups which have Square-Integrable Densities

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**Abstract**

We apply Peter-Weyl theory to obtain necessary and sufficient conditions for a probability measure on a compact group to have a square-integrable density. Applications are given to measures on the $d$-dimensional torus.

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## 1 Introduction

Given a probability measure on a Riemannian manifold it is highly beneficial if we can establish that it has a density with respect to the Riemannian volume measure. This is vital for statistical inference but a density may also contain important information about the topology and geometry of the manifold, e.g. the heat kernel is the density of Brownian motion on a manifold (see e.g. [6], [18].) Much work by stochastic analysts has been focussed on finding densities for solutions of stochastic differential equations driven by Brownian motion. A key result here is Hörmander's theorem [10] which gives a sufficient condition (called "hypoellipticity") for existence of a smooth density involving the Lie algebra generated by the driving vector fields. The search for equivalent probabilistic conditions for hypoellipticity led to the Malliavin calculus [17]. Ongoing research is now developing this into a tool to investigate densities of more sophisticated systems such as solutions of stochastic differential equations driven by non-Gaussian noise (see e.g. [11]) and solutions of stochastic partial differential equations (see e.g. [19] and references therein.)

In this short note we focus on the case of a compact group $G$. Recently Liao (see Chapter 4 in [13] or [14]) has established existence of square-integrable densities for $G$-valued Lévy processes (i.e. processes with independent and stationary increments) in the case where $G$ is a Lie group and the laws are either conjugate invariant or invariant under the inverse map. His approach makes extensive use of Fourier analysis - especially the Peter Weyl theorem - and requires the hypoellipticity assumption to hold for the diffusion part. More recently, working together with L.Wang [15], he has obtained densities of convolution semigroups of probability measures on symmetric spaces. The main purpose of this note is to adapt Liao's approach to arbitrary probability measures on a compact group. We give four equivalent necessary and sufficient conditions for such a measure to have a square-integrable density with respect to Haar measure. These effectively require summability of squares of matrix elements taken over all irreducible representations of the associated non-commutative Fourier transform (for background on which see [12], section 3.2. in [7], [8], [21] or [2].) This can to some extent be seen as analogous to the well-known result that a probability measure on Euclidean space has a density if its Fourier transform (characteristic function) is absolutely integrable (see e.g. Theorem 3.2.2 in [16].) As an application of our result we are able to show that a large class of infinitely divisible probability measures on the $d$-dimensional torus $\Pi^d$ have an $L^2$-density if they have a non-degenerate Gaussian component. In fact this condition is equivalent to hypoellipticity and this result, together with those of Liao which were discussed above leads us to conjecture that hypoellipticity is a sufficient condition for the existence of a density for an arbitrary infinitely divisible probability measure on a compact Lie group. In the last part of the paper we give some examples of infinitely divisible measures on $\Pi^d$ which have no Gaussian component but still have an $L^2$-density.

*Notation.* If $T$ is a bounded linear operator acting in a complex Hilbert space $H$ then $T^*$ denotes its adjoint. When $H$ is finite-dimensional then $T$ may be identified with a complex matrix and $T^*$ is the complex conjugate of the transpose of $T$. If $z$ is a complex number then $\Re(z)$ and $\Im(z)$ are its real and imaginary parts (respectively).

# 2 Preliminaries

Let $G$ be a locally compact group with neutral element $e$ and let $\mathrm{Rep}(G)$ be the set of all equivalence classes of unitary representations of $G$ in a complex Hilbert space. $\mathrm{Irr}(G)$ is the subset obtained from all irreducible representations and those that are non-trivial are denoted by $\mathrm{Irr}_+(G)$. We will, as usual, identify each equivalence class with a representative element. If $\pi \in \mathrm{Rep}(G)$, then for each $g \in G, \pi(g)$ is a unitary operator in the Hilbert space $V_\pi$. The $\sigma$-algebra of all Borel sets in $G$ is denoted by $\mathcal{B}(G)$. Let $\mathcal{M}(G)$

denote the set of all Borel probability measures on $G$. It is a monoid with respect to the binary operation of convolution where the identity element is the Dirac mass at $e$. We recall that the convolution $\mu_1 * \mu_2$ of $\mu_1, \mu_2 \in \mathcal{M}(G)$ is the unique Borel probability measure for which

$$\int_G f(\sigma)(\mu_1 * \mu_2)(d\sigma) = \int_G \int_G f(\sigma\tau)\mu_1(d\sigma)\mu_2(d\tau),$$

for all real-valued bounded Borel functions $f$ defined on $G$. The *reversed measure* $\widetilde{\mu}$ that is associated to each $\mu \in \mathcal{M}(G)$ is defined by $\widetilde{\mu}(A) = \mu(A^{-1})$ for each $A \in \mathcal{B}(G)$. Note that $\widetilde{\mu} \in \mathcal{M}(G)$ and that $\widetilde{\mu_1 * \mu_2} = \widetilde{\mu_2} * \widetilde{\mu_1}$ for each $\mu_1, \mu_2 \in \mathcal{M}(G)$. $\mu \in \mathcal{M}(G)$ is said to be symmetric if $\widetilde{\mu} = \mu$. In particular each of $\widetilde{\mu} * \mu$ and $\mu * \widetilde{\mu}$ are symmetric for arbitrary $\mu \in \mathcal{M}(G)$, but these measures may be distinct if $G$ is not abelian.

Fix $\mu \in \mathcal{M}(G)$. The *non-commutative Fourier transform* of $\mu$ is defined as a Bochner integral in the Banach space of bounded linear operators on $V_\pi$ by the prescription

$$\widehat{\mu}(\pi) = \int_G \pi(g)\mu(dg),$$

for each $\pi \in \mathrm{Rep}(G)$. It is easy to see that

(F1) $\widehat{\mu}(\pi)$ is a contraction on $V_\pi$,

(F2) $\widehat{\widetilde{\mu}}(\pi) = \widehat{\mu}(\pi)^*$,

(F3) $\widehat{\mu_1 * \mu_2}(\pi) = \widehat{\mu_1}(\pi)\widehat{\mu_2}(\pi)$, for each $\mu_1, \mu_2 \in \mathcal{M}(G)$.

For explicit proofs and further investigations into these ideas see [8], [21], [2].

# 3 Densities on Compact Groups

From now on we will take $G$ to be compact. In this case for each $\pi \in \mathrm{Irr}(G), d_\pi := \dim(V_\pi) < \infty$. Furthermore any Haar measure defined on $G$ is finite and bi-invariant. We choose normalised Haar measure and use the standard notation $dg$ to denote its "differential". The corresponding $L^2$ space is denoted by $L^2(G, \mathbb{C})$. The celebrated Peter-Weyl theorem tells us that $\{d_\pi^{\frac{1}{2}}\pi_{ij}; 1 \leq i, j \leq d_\pi, \pi \in \mathrm{Irr}(G)\}$ is a complete orthonormal basis for

$L^2(G,\mathbb{C})$ where the co-ordinate functions $\pi_{ij}$ are defined by the prescription $\pi_{ij}(g) := \pi(g)_{ij}$, for each $1 \leq i, j \leq d_\pi$. Hence if $f \in L^2(G,\mathbb{C})$,

$$f = \langle f \rangle + \sum_{\pi \in \mathrm{Irr}_+(G)} d_\pi \sum_{i,j=1}^{d_\pi} \langle f, \pi_{ij} \rangle \pi_{ij}, \tag{3.1}$$

where $\langle f \rangle := \int_G f(\sigma) d\sigma$. Straightforward algebraic manipulation yields the following alternative representation (see e.g. [4]):

$$f = \langle f \rangle + \sum_{\pi \in \mathrm{Irr}_+(G)} d_\pi \mathrm{tr}(\widehat{f}(\pi)\pi), \tag{3.2}$$

where $\widehat{f}(\pi) := \int_G f(g)\pi(g^{-1})dg$.

We say that $\mu \in \mathcal{M}(G)$ has a *square-integrable density* if $\mu$ is absolutely continuous with respect to Haar measure and the Radon-Nikodým derivative $\frac{d\mu}{dg}$ lies in the space $L^2(G,\mathbb{R})$. If this is the case, we observe that if we write $f := \frac{d\mu}{dg}$, then

$$\widehat{f}(\pi) = \int_G \pi(g^{-1})\mu(dg) = \int_G \pi(g)\widetilde{\mu}(dg) = \widehat{\widetilde{\mu}}(\pi).$$

An easy consequence of (3.1) and Parseval's identity is that

$$||f||^2 = 1 + \sum_{\pi \in \mathrm{Irr}_+(G)} d_\pi \sum_{i,j=1}^{d_\pi} |\widehat{\mu}(\pi)_{ij}|^2. \tag{3.3}$$

The main result of this paper is the following:

**Theorem 3.1** *$\mu \in \mathcal{M}(G)$ has a square-integrable density if and only if any one of the following equivalent conditions is satisfied.*

*(i)* $\sum_{\pi \in Irr_+(G)} d_\pi \sum_{i,j=1}^{d_\pi} |\widehat{\mu}(\pi)_{ij}|^2 < \infty$,

*(ii)* $\sum_{\pi \in Irr_+(G)} d_\pi tr(\widehat{\mu}(\pi)^* \widehat{\mu}(\pi)) < \infty$,

*(iii)* $\sum_{\pi \in Irr_+(G)} d_\pi tr(\widehat{\widetilde{\mu} * \mu}(\pi)) < \infty$.

*(iv)* $\sum_{\pi \in Irr_+(G)} d_\pi tr(\widehat{\mu * \widetilde{\mu}}(\pi)) < \infty$.

*If this is the case we have*

$$\frac{d\mu}{dg} = 1 + \sum_{\pi \in Irr_+(G)} d_\pi tr(\widehat{\mu}(\pi)^* \pi). \tag{3.4}$$

*Proof.* The equivalence of (i) and (ii) is easy linear algebra while that of (ii) and (iii) is straightforward from (F2) and (F3). That of (iii) and (iv) follows similarly since for each $\pi \in \mathrm{Irr}_+(G)$,

$$\mathrm{tr}(\widehat{\mu * \widetilde{\mu}}(\pi)) = \mathrm{tr}(\widehat{\mu}(\pi)\widehat{\mu}(\pi)^*) = \mathrm{tr}(\widehat{\mu}(\pi)^*\widehat{\mu}(\pi)) = \mathrm{tr}(\widehat{\widetilde{\mu} * \mu}(\pi)).$$

Necessity follows from (3.3). For sufficiency we define $f := 1+\sum_{\pi\in\mathrm{Irr}_+(G)} d_\pi \mathrm{tr}(\widehat{\mu}(\pi)^*\pi)$, so that $f \in L^2(G,\mathbb{C})$. By uniqueness of the Peter-Weyl expansion we have $\widehat{f}(\pi) = \widehat{\mu}(\pi)^*$.

By the Peter-Weyl theorem, if $g \in C(G,\mathbb{C})$ then its Fourier series (as given by (3.1) or (3.2)) converges uniformly (see e.g. [4]). Hence by Plancherel's theorem for each $g \in C(G,\mathbb{C})$,

$$\begin{aligned}
\int_G g(\sigma)\overline{f(\sigma)}d\sigma &= \sum_{\pi\in\mathrm{Irr}(G)} d_\pi \sum_{i,j=1}^{d_\pi} \langle g, \pi_{ij}\rangle\langle \pi_{ij}, f\rangle \\
&= \sum_{\pi\in\mathrm{Irr}(G)} d_\pi \mathrm{tr}(\widehat{g}(\pi)\widehat{f}(\pi)^*) \\
&= \sum_{\pi\in\mathrm{Irr}(G)} d_\pi \mathrm{tr}(\widehat{g}(\pi)\widehat{\mu}(\pi)).
\end{aligned}$$

However by (3.2) and Fubini's theorem

$$\begin{aligned}
\int_G g(\sigma)\mu(d\sigma) &= \int_G \sum_{\pi\in\mathrm{Irr}(G)} d_\pi \mathrm{tr}(\widehat{g}(\pi)\pi(\sigma))\mu(d\sigma) \\
&= \sum_{\pi\in\mathrm{Irr}(G)} d_\pi \mathrm{tr}(\widehat{g}(\pi)\widehat{\mu}(\pi)).
\end{aligned}$$

It follows that

$$\int_G g(\sigma)\overline{f(\sigma)}d\sigma = \int_G g(\sigma)\mu(d\sigma). \tag{3.5}$$

Consequently we have $\int_G g(\sigma)\Im(f)(\sigma)d\sigma = 0$ for all $g \in C(G,\mathbb{C})$. Since $C(G,\mathbb{C})$ is dense in $L^2(G,\mathbb{C})$, it follows that $\Im(f) = 0$ (a.e.). Now assume that $B := f^{-1}((-\infty,0))$ has positive Haar measure and choose $g \in C(G,\mathbb{R})$ to be strictly negative with $\mathrm{supp}(g) \subseteq B$. Note that such a $g$ can always be found since Haar measure is regular. We then obtain

$$0 < \int_B g(\sigma)f(\sigma)d\sigma = \int_G g(\sigma)f(\sigma)d\sigma = \int_G g(\sigma)\mu(d\sigma) = \int_B g(\sigma)\mu(d\sigma) \leq 0.$$

We have derived a contradiction and hence can deduce that $f \geq 0$ (a.e.) We now apply the Riesz representation theorem (see e.g. Theorem 7.2.8 in [5]) to both sides of (3.5) to deduce that $f(\sigma)d\sigma = \mu(d\sigma)$ as was required. □

# 4 Infinitely Divisible Measures on the $d$-dimensional Torus

We recall that a probability measure on a locally compact group $G$ is *infinitely divisible* if it has a convolution $n$th root for all $n \in \mathbb{N}$. We denote the set of all such measures by $\mathrm{ID}(G)$. It is well known (see e.g. [1], [3], [20]) that there is a one-to-one correspondence between $\mathrm{ID}(\mathbb{R}^d)$ and the set of all continuous, hermitian, negative definite functions $\eta : \mathbb{R}^d \to \mathbb{C}$ for which $\eta(0) = 0$ via the prescription

$$\int_{\mathbb{R}^d} e^{iu\cdot x}\mu(dx) = e^{-\eta(u)},$$

for each $u \in \mathbb{R}^d$. The generic form of $\eta$ is given by the Lévy-Khinchine formula

$$\eta(u) = ib\cdot u + \frac{1}{2}u\cdot Au + \int_{\mathbb{R}^d-\{0\}}(1 - e^{iu\cdot y} + iu\cdot y 1_{B_1}(y))\nu(dy), \tag{4.6}$$

where $b \in \mathbb{R}^d, A$ is a non-negative definite symmetric $d \times d$ matrix, $\nu$ is a Borel measure on $\mathbb{R}^d - \{0\}$ for which $\int_{\mathbb{R}^d-\{0\}}(|y|^2 \wedge 1)\nu(dy) < \infty$ and $B_1$ is the open ball of radius one centred at the origin. The triple $(b, A, \nu)$ is called the characteristics of $\mu$. $\mu \in \mathcal{M}(\mathbb{R}^d)$ is Gaussian (in the usual sense) if and only if $\mu \in \mathrm{ID}(\mathbb{R}^d)$ with characteristics $(b, A, 0)$ and more generally an arbitrary $\mu \in \mathrm{ID}(\mathbb{R}^d)$ with characteristics $(b, A, \nu)$ has a *non-degenerate Gaussian component* if $A$ is non-singular. Now consider the $d$-dimensional torus $\Pi^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ and let $\gamma : \mathbb{R}^d \to \Pi^d$ be the canonical surjection. If $\mu \in \mathrm{ID}(\mathbb{R}^d)$ then it is easily verified that $\mu^\gamma \in \mathrm{ID}(\Pi^d)$ where $\mu^\gamma := \mu \circ \gamma^{-1}$. Any $\rho \in \mathrm{ID}(\Pi^d)$ which arises in this way will be called *projective*. There are probability measures on $\Pi^d$ which are infinitely divisible but not projective e.g. normalised Haar measure. If $\mu$ has a density $f$ then it is easily verified that $\mu^\gamma$ has a density $f^\gamma$ where for all $x \in \Pi^d$,

$$f^\gamma(x) = \sum_{n\in\mathbb{Z}^d} f(x + 2\pi n). \tag{4.7}$$

The irreducible representations on $\Pi^d$ are precisely the characters $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and for each $\mu \in \mathrm{ID}(\mathbb{R}^d), n \in \mathbb{Z}^d$ we have

$$\widehat{\mu^\gamma}(n) = \int_{\Pi^d} e^{in\cdot x}\mu^\gamma(dx) = \int_{\mathbb{R}^d} e^{in\cdot y}\mu(dy) = e^{-\eta(n)}. \tag{4.8}$$

**Proposition 4.1** *If $\mu \in ID(\mathbb{R}^d)$ has a non-degenerate Gaussian component then $\mu^\gamma$ has an $L^2$ density.*

*Proof.* Using condition (i) in Theorem 3.1 and (4.8), we see that we must show that

$$\sum_{n \in \mathbb{Z}^d} |e^{-\eta(n)}|^2 = \sum_{n \in \mathbb{Z}^d} e^{-2\Re(\eta(n))} < \infty.$$

Now by (4.6)

$$\begin{aligned} \Re(\eta(n)) &= \frac{1}{2} n \cdot An + \int_{\mathbb{R}^d - \{0\}} (1 - \cos(n \cdot y))\nu(dy) \\ &\geq \frac{1}{2} n \cdot An, \end{aligned}$$

and so

$$\sum_{n \in \mathbb{Z}^d} |e^{-\eta(n)}|^2 \leq \sum_{n \in \mathbb{Z}^d} e^{-n \cdot An} \leq \sum_{n \in \mathbb{Z}^d} e^{-\lambda n \cdot n} < \infty,$$

where $\lambda > 0$ is the minimum eigenvalue of $A$. □

**Remarks**

1. If $\mu$ satisfies the hypothesis of Proposition 4.1, then it is easily verified that its characteristic function is absolutely integrable and so $\mu$ has a density. Consequently the fact that $\mu^\gamma$ has a density is immediate from (4.7). The new information that we have gained from Proposition 4.1 is the square integrability and (3.4) is then the usual Fourier series expansion.

2. If $\mu$ is a standard Gaussian measure (i.e. $A = I$), then it is well-known that $\mu^\gamma$ is a product of Jacobi theta functions - indeed this is easily verified from (4.7) by using the Poisson summation formula (see e.g. [9], p.375).

3. The condition that the Gaussian component of $\mu$ is non-degenerate is easily seen to be equivalent to hypoellipticity of the vector fields $\{Y_1, \ldots, Y_d\}$, where $Y_i = \sum_{j=1}^d A_{ij}\partial_j$, for $1 \leq i \leq d$. Indeed an alternative proof of Proposition 4.1 could be obtained by making use of this fact and applying Lemma 4.1 in [13].

For an example of a family of infinitely divisible probability measures on $\Pi^1$ each of which have an $L^2$ density but with no Gaussian component (i.e. $A = 0$), we take $\mu \in \mathrm{ID}(\mathbb{R})$ to be any $\alpha$-stable distribution for which $0 < \alpha < 2$. In this case there exists $\sigma > 0$ such that (see e.g. Theorem 14.15 in [20])

$$\sum_{n\in\mathbb{Z}} |e^{-\eta(n))}|^2 = \sum_{n\in\mathbb{Z}} e^{-2\sigma|n|^\alpha} = 1 + 2\sum_{n=1}^{\infty} e^{-2\sigma n^\alpha}. \tag{4.9}$$

The convergence of this series follows by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ since $\lim_{n\to\infty} n^2 e^{-2\sigma n^\alpha} = \lim_{x\to\infty} x^{\frac{2}{\alpha}} e^{-2\sigma x} = 0$.

Now suppose that for $0 < \alpha < 2$, $\mu_\alpha$ is a rotationally invariant (non-Gaussian) $\alpha$-stable distribution on $\mathbb{R}^d$. Then (4.6) can be written in the form $\eta_\alpha(u) = c||u||^\alpha$ for all $u \in \mathbb{R}^d$, where $c > 0$ and $||\cdot||$ denotes the Euclidean norm. For later reference we note that $\mu_\alpha$ has characteristics $(0, 0, \nu_\alpha)$ where $\nu_\alpha(dx) = \frac{K}{||x||^{\alpha+d}}$ for some $K > 0$ (see e.g. section 3.14 in [20].) We can now show that $\mu_\alpha^\gamma$ has an $L^2$-density on $\Pi^d$, indeed by Hölder's inequality we have

$$\sum_{n\in\mathbb{Z}^d} e^{-2c||n||^\alpha} \leq \sum_{n\in\mathbb{Z}^d} \exp\left\{-\kappa\sum_{i=1}^{d} |n_i|^\alpha\right\} = \left(\sum_{m\in\mathbb{Z}} e^{-\kappa|m|^\alpha}\right)^d < \infty \tag{4.10}$$

by the remarks following (4.9) where $\kappa := 2cd^{\frac{2}{2-\alpha}}$.

We close this paper with a result that allows us to obtain $L^2$-densities for a wider class of infinitely divisible distributions on $\Pi^d$. First we need a definition. Let $\nu_1$ and $\nu_2$ be Lévy measures on $\mathbb{R}^d - \{0\}$. We say that $\nu_1$ *dominates* $\nu_2$ if there exists $k > 0$ such that

$$\nu_1(U) \geq k\nu_2(U)$$

for all Borel sets $U$ which are bounded away from zero in the sense that $0 \notin \overline{U}$. For a simple example we take $\nu_2 \equiv \nu_\alpha$ and $\nu_1(dx) = h(x)\nu_\alpha(dx)$ where $h(x) = \begin{cases} e^{c||x||} & \text{if } ||x|| < 1 \\ 1 & \text{if } ||x|| \geq 1 \end{cases}$ and $c > 0$. In this case we may take $0 < k \leq 1$.

The idea for the following result was suggested to the author by Ming Liao.

**Theorem 4.1** *Let $\mu$ be an infinitely divisible distribution on $\mathbb{R}^d$ with characteristics $(b, A, \nu)$. If $\nu$ dominates $\nu_\alpha$ for some $0 < \alpha < 2$ then $\mu^\gamma$ has a square-integrable density on $\Pi^d$.*

*Proof.* Let $(U_m, m \in \mathbb{N})$ be an increasing sequence of Borel sets that are bounded away from zero with $U_m \uparrow \mathbb{R}^d - \{0\}$ as $m \to \infty$.

Using (4.6) we have for each $m \in \mathbb{N}, n \in \mathbb{Z}^d$,

$$\begin{aligned}\Re(\eta(n)) &= \frac{1}{2} n \cdot An + \int_{U_m} (1 - \cos(n \cdot x))\nu(dx) + \int_{\mathbb{R}^d - \{0\} - U_m} (1 - \cos(n \cdot x))\nu(dx) \\ &\geq \int_{U_m} (1 - \cos(n \cdot x))\nu(dx) \\ &\geq k \int_{U_m} (1 - \cos(n \cdot x))\nu_\alpha(dx).\end{aligned}$$

Taking limits as $m \to \infty$, we deduce that

$$\Re(\eta(n)) \geq k \int_{\mathbb{R}^d - \{0\}} (1 - \cos(n \cdot x))\nu_\alpha(dx) = k\eta_\alpha(n).$$

The required result now follows from (4.10). $\square$

We remark that the verification of Theorem 3.1(i) for the case of measures on $SU(2)$ may at least in principle be carried out using the fact that all matrix elements of irreducible representations can be expressed in terms of Jacobi polynomials (see e.g. [22], p.97).

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