MAS221: Characterising Completeness of the Real Number Line

This material is non-examinable. It would be best to read it after completing Chapter 2 of the notes.

In this document, we will look at three different ways of characterising completeness of the real number line, and prove that they are all equivalent. These are

1. The *completeness property* which says that every non-empty set of real numbers that is bounded above has a least upper bound.

2. The *nested interval principle* Given a sequence of closed intervals $([a_n, b_n], n \in \mathbb{N})$, which are nested in that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$, with $\lim_{n\to\infty}(b_n - a_n) = 0$, then there exists a unique $x \in \mathbb{R}$ so that $x \in [a_n, b_n]$ for all $n \in \mathbb{N}$.

3. Every Cauchy sequence converges.

First we need a lemma:

Lemma 0.0.1. *Let $A \subset \mathbb{R}$ be a non-empty set that is bounded above. Suppose there exists $x \in \mathbb{R}$ such that $x + \epsilon$ is an upper bound for A for all $\epsilon > 0$. Then x is also an upper bound for A.*

Proof. Suppose that x is not an upper bound for A. Then there exists $y \in A$ for which $x < y$. But then $x' = x + \frac{1}{2}(y - x)$ is of the form $x + \epsilon$, with $\epsilon > 0$, but $x' < y$, and this yields the required contradiction. □

Theorem 0.0.2. *The following three properties of the real numbers are equivalent:*

(i) The nested interval principle,

(ii) The completeness property,

(iii) Every Cauchy sequence converges.

Proof. (i) $\Rightarrow$ (ii). Let S be a non–empty set of real numbers that is bounded above. Let $a_1 \in \mathbb{R}$ be such that a_1 is not an upper bound for S (e.g. choose $a \in S$ and $a_1 = a - \epsilon$, where $\epsilon > 0$). Let b_1 be an upper bound for S, so $a_1 < b_1$. Define $c_1 = (a_1 + b_1)/2$. If c_1 is an upper bound for S, choose $b_2 = c_1$ and $a_2 = a_1$. If c_1 is not an upper bound for S, choose $b_2 = b_1$ and $a_2 = c_1$. Either way, we have $[a_2, b_2] \subset [a_1, b_1]$. Now iterate this procedure, i.e. for $k = 2, 3, 4, \ldots$, once we have found a_k and b_k, define $c_k = (a_k + b_k)/2$.

If c_k is an upper bound for S, choose $b_{k+1} = c_k$ and $a_{k+1} = a_k$. If c_k is not an upper bound for S, choose $b_{k+1} = b_k$ and $a_{k+1} = c_k$. Then we have a sequence $([a_n, b_n], n \in \mathbb{N})$ of nested intervals and for all $n \in \mathbb{N}$

$$b_{n+1} - a_{n+1} = (b_n - a_n)/2 = \cdots = (b_1 - a_1)/2^n \to 0 \text{ as } n \to \infty.$$

So by the nested interval principle, there exists $x \in \mathbb{R}$ so that $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$.

Given any $y < x$, we can find $m \in \mathbb{N}$ so that $y < a_m$, for otherwise $a_n \leq y < x \leq b_n$ for all $n \in \mathbb{N}$, and this contradicts the uniqueness of x. By construction of the sequence of a_n's, a_m cannot be an upper bound for S, and so neither can y. Given any $z > x$, we can find $k \in \mathbb{N}$ so that $b_k \leq z$, for otherwise we contradict uniqueness of x, as above. By construction of the sequence of b_n's, b_k is an upper bound for S and hence so is z. Since every number larger than x is an upper bound for S, and every number smaller than x is not an upper bound, we conclude firstly using Lemma 0.0.1 that x is an upper bound, and secondly that it is the least upper bound.

(ii) $\Rightarrow$ (iii). Let $(a_n, n \in \mathbb{N})$ be Cauchy and define the set $S_1 = \{a_1, a_2, a_3, \ldots\}$. Since every Cauchy sequence is bounded (by Problem 41), S_1 is a bounded (non-empty) set of real numbers, and so it has a greatest lower bound L_1. Now consider the set $S_2 = \{a_2, a_3, a_4, \ldots\}$, i.e. $S_2 = S_1 \setminus \{a_1\}$. Then S_2 is bounded and its greatest lower bound $L_2 \geq L_1$. We continue in this fashion, defining $S_{k+1} = S_k \setminus \{a_k\}$ and $L_{k+1} = \inf(S_{k+1})$ for $k = 2, 3, \ldots$. In this way we obtain a sequence (L_n) which is monotonic increasing and bounded above (indeed, any upper bound for S is also a bound for (L_n)). Then (L_n) converges by Theorem 2.3.1 and we write[1] $M = \lim_{n\to} L_n$. We will show that $M = \lim_{n\to\infty} a_n$ and then (iii) is established.

By Proposition 1.4.2, since $M = \sup(L_n)$, given any $\epsilon > 0$, there exists $K \in \mathbb{N}$ so that $L_K > M - \epsilon$. Then since (L_n) is monotonic increasing $L_n > M - \epsilon$ for all $n \geq K$. Now since (a_n) is Cauchy, there exists $N_0 \in \mathbb{N}$ so that for all $n, m > N_0, |a_n - a_m| < \frac{\epsilon}{2}$. Next let $N \geq \max\{K, N_0\}$. Since $L_N = \inf\{a_N, a_{N+1}, a_{N+2}, \ldots\}$, by Problem 15(b), there exists $m > N$ so that $a_m < L_N + \epsilon/2$.

Now we put all the pieces together. Given our $\epsilon > 0$, for all $n \geq N$,

$$M - \epsilon < L_K \leq L_N \leq a_n < a_m + \epsilon/2 < L_N + \epsilon \leq M + \epsilon,$$

i.e. $M - \epsilon < a_n < M + \epsilon$, and so $M = \lim_{n\to\infty} a_n$.

[1]In fact, $M = \liminf_{n\to\infty} a_n$ in the language of Problem 40

(iii) $\Rightarrow$ (i).

Let $([a_n, b_n], n \in \mathbb{N})$ be a sequence of nested intervals. We need to show that there is a unique $x \in [a_n, b_n]$ for all $n \in \mathbb{N}$. First observe that (a_n) is a Cauchy sequence. Indeed since $\lim_{n\to\infty}(b_n - a_n) = 0$, given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that $b_n - a_n < \epsilon$. But for all $m > N$,

$$a_m - a_n \leq b_n - a_n < \epsilon,$$

and so (a_n) is Cauchy. Let $x = \lim_{n\to\infty} a_n$. Since the sequence (a_n) is monotonic increasing, and bounded above (e.g. by b_1), $x = \sup_{n\in\mathbb{N}} a_n \geq a_m$ for all $m \in \mathbb{N}$. We need to show that $x \leq b_n$ for all $n \in \mathbb{N}$, Suppose that we can find $k \in \mathbb{N}$ so that $b_k < x$. Choose $0 < \epsilon < x - b_k$. Since $x = \lim_{n\to\infty} a_n$, we can find $N \in \mathbb{N}$ so that $n > N \Rightarrow x - a_n < \epsilon$. But then $b_k < a_k$ and we have a contradiction. Now suppose that $y \neq x$ also satisfies $y \in [a_n, b_n]$ for all $n \in \mathbb{N}$. Either $y < x$ or $y > x$. If $y < x$ then y is an upper bound for (a_n), which is smaller than $x = \sup_{n\in\mathbb{N}} a_n$ and that yields a contradiction. Now suppose $y > x$. By similar arguments to the above we can show that $x = \lim_{n\to\infty} b_n = \inf_{n\in\mathbb{N}} b_n$, and we again obtain a contradiction.

□

If $(A_n, n \in \mathbb{N})$ is a sequence of subsets of a set A, define

$$\bigcap_{n\in\mathbb{N}} A_n = \{a \in A; a \in A_n \text{ for all } n \in \mathbb{N}\}.$$

Then, under the given hypotheses, the nested interval principle tells us that $\bigcap_{n\in\mathbb{N}}[a_n, b_n] \neq \emptyset$. This would not necessarily be true if we replaced closed intervals with open intervals, e.g. $\bigcap_{n\in\mathbb{N}}(0, 1/n) = \emptyset$.