MAS221 Analysis – Exercises II

(*Problems labelled * may be more demanding.*)

Problems for Chapter 3

44. Write down the domains of the following mappings from $\mathbb{R}$ to $\mathbb{R}$:

    (a) $f_1(x) = \dfrac{x^2+2x+7}{x(x+1)}$,

    (b) $f_2(x) = \dfrac{(x-1)(x+4)}{x^3+4x^2+x-6}$,

    (c) $f_3(x) = \dfrac{x+4}{x^2+5x+6}$,

    (d) $f_4(x) = \exp\left\{-\dfrac{1}{x-1}\right\}$,

    (e) $f_5(x) = \cos\left(\dfrac{1}{\pi x}\right)$.

45. Investigate $\lim_{x\to 1} f_2(x), \lim_{x\to -2} f_2(x)$ and $\lim_{x\to -3} f_2(x)$, and calculate the value of the limit, when it exists.

46. If $f : \mathbb{R} \to [0,\infty)$ satisfies $\lim_{x\to a} f(x) = l$, where $l > 0$, show that $\lim_{x\to a} \sqrt{f(x)} = \sqrt{l}$. Hence calculate $\lim_{x\to 1} \sqrt{\dfrac{x+1}{x^2}}$.

47. Why are limits of functions unique (compare with Theorem 2.1.1)?

48. Verify that $\operatorname{sgn}(x) = \dfrac{|x|}{x} = \dfrac{x}{|x|}$, for $x \neq 0$ (see Example 3.3 in the notes for the definition). Deduce that $\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist. Show that both the left and right limits exist at $x = 0$, and find their values.

49. For the following functions, each of which is defined on the whole of $\mathbb{R}$, find every point at which both the left and right limits exist, and are different, and find the values of these limits:

    (a) $f(x) = \begin{cases} 1-x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1, \end{cases}$

    (b) $g(x) = [x]$, where $[x]$ is the integer part of x, i.e. the greatest integer less than or equal to x.

    (c) $h(x) = 3 - 5\mathbf{1}_{(0,1]}(x) + 7\mathbf{1}_{(1,2]}(x)$.

50. Consider the function $f(x) = \sin(1/x)$. What is its domain? Does it have a limit at $x = 0$? [Hint: Consider sequences whose nth term is $1/(\theta + 2n\pi)$, and think about good choices for θ.]

51. Consider the function $f(x) = x\sin(1/x)$. What is its domain? Does it have a limit at $x = 0$?

52. Prove Theorem 3.3.1 in the notes to establish $\epsilon - \delta$ criteria for left and right limits.

53. In the notes, we gave meaning to $\lim_{x\to a} f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is a function and $a \in \mathbb{R}$. In this question, you can investigate what happens when $a = \infty$ and $a = -\infty$.

    (a) Formulate a rigorous definition, in terms of sequences, for $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to exist.

    (b) Find an analogue of the $(\epsilon - \delta)$ criterion for this case, and prove the analogous result to Theorem 3.2.2.

    (c) Check that you are able to prove that $\lim_{x\to\infty} 1/x = \lim_{x\to-\infty} 1/x = 0$, using your criterion.

54. (a) Formulate a rigorous definition, in terms of sequences, for $\lim_{x\to\infty} f(x) = \pm\infty$ and $\lim_{x\to-\infty} f(x) = \pm\infty$ to exist, and write down an analogue of the $(\epsilon - \delta)$ criterion.

    (b) Let $f : \mathbb{R} \to [0, \infty)$ and $g : \mathbb{R} \to [0, \infty)$ both have domain $\mathbb{R}$, and suppose that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = l$, where $l > 0$. Define $h(x) = f(x)g(x)$ for all $x \in \mathbb{R}$. Show that $\lim_{x\to\infty} h(x) = \infty$. What happens if all instances of ∞ in the above are replaced with $-\infty$?

    (c) Let $p : \mathbb{R} \to \mathbb{R}$ be an even polynomial of degree m, where the leading coefficient (i.e. the coefficient of x^m) is positive. Show that $\lim_{x\to\infty} p(x) = \lim_{x\to-\infty} p(x) = \infty$. What happens when m is odd?

Problems for Chapter 4

55. Return to Problem 44. For each function considered there, what is the maximum subset of its domain on which it is continuous?

56. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at a point a.

i. Prove that $|f|$ is continuous at a, where $|f|(x) = |f(x)|$, for all $x \in D_f$.

ii. Assuming that $f(a) \neq 0$, show that $\sqrt{|f|}$ is continuous at a, where $\sqrt{|f|}(x) = \sqrt{|f(x)|}$, for all $x \in D_f$.

(b) Find the maximum subset of $[0, \infty)$ for which the mapping $x \to \sqrt{x}$ is continuous.

57. Prove Theorem 4.1.3 in the notes (i.e. that composition of continuous functions is continuous). [Hint: Use sequences.]

58. Define f and $g : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/x$ for $x \in \mathbb{R}\backslash\{0\}$, and $g(x) = 1+x^2$ for $x \in \mathbb{R}$. Write down the functions $f \circ g$, and $g \circ f$, giving their domains explicitly. What can you say about continuity of these functions?

59. For each of the functions in Problem 49,

(a) Find the maximum subset of its domain on which it is continuous.

(b) Identify those discontinuities which are jumps, and calculate the size of each jump.

60. (a) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{(1+x)^2 - 1}{x}$$

with $D_f = \mathbb{R} \setminus \{0\}$. Find an extension of f to the whole of $\mathbb{R}$ that is continuous there.

(b) Write down the domain D_f of the function $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$ and explain why f is continuous at every point of D_f. The complement D_f^c of D_f in $\mathbb{R}$ comprises two points. Show that f may be extended to be continuous at only one of these points, and write down this continuous extension.

61. Suppose that the mapping $g : \mathbb{R} \to \mathbb{R}$ is continuous at a with $g(a) > 0$ (or alternatively, $g(a) < 0$). Show that there exists $\delta > 0$ so that $g(x) > 0$ (respectively, $g(x) < 0$) for all $x \in (a - \delta, a + \delta)$. [Hint: Assume $g(a) > 0$ and try a proof by contradiction.]

62. (a) For any $x, y \in \mathbb{R}$ show that

$$\max\{x, y\} = \frac{1}{2}(x + y) + \frac{1}{2}|x - y|.$$

Hence show that if both f and $g : \mathbb{R} \to \mathbb{R}$ are continuous at a, then so is $\max\{f, g\}$, where for all $x \in D_f \cap D_g$,

$$\max\{f, g\}(x) = \max\{f(x), g(x)\}.$$

(b) Find a similar expression for $\min\{x, y\}$, and hence prove continuity of $\min\{f, g\}$ at a.

63. The aim of this question is to prove the following: the only possible functions $f : \mathbb{R} \to \mathbb{R}$ with domain $\mathbb{R}$ which are continuous at zero and satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ are the linear mappings $f(x) = kx$ for all $x \in \mathbb{R}$, where $k \in \mathbb{R}$ is fixed.

    Begin by considering $f : \mathbb{R} \to \mathbb{R}$ with domain $\mathbb{R}$, such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

    (a) Prove that $f(0) = 0$,
    (b) Show that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$,
    (c) If f is continuous at zero, prove that it is continuous at every $x \in \mathbb{R}$,
    (d) If $f(1) = k$, prove that $f(n) = kn$ for all $n \in \mathbb{Z}$,
    (e) If $f(1) = k$, prove that $f(p/q) = kp/q$ for all $p/q \in \mathbb{Q}$,
    (f) If $f(1) = k$ and f is continuous at zero, prove that $f(x) = kx$ for all $x \in \mathbb{R}$.

64. Show that Dirichlet's "other" function, as discussed in Example 4.5 in the notes, is discontinuous at every rational point in its domain.

65. What can you say about left/right continuity of the mapping $x \to \mathbf{1}_{(a,b)}$ at the points a and b?

66. Prove Corollary 4.3.2. [Hint: Apply the intermediate value theorem to the function $g(x) = f(x) - \gamma$, for $x \in [a, b]$.]

67. Use Corollary 4.3.2 to show that

    (a) Every continuous function from $\mathbb{R}$ to $\mathbb{Z}$ is constant,
    (b) Every continuous function from $\mathbb{R}$ to $\mathbb{Q}$ is constant.

68. Prove the following *fixed point theorem*: if $f : [a, b] \to (a, b)$ is continuous, then there exists $c \in (a, b)$ such that $f(c) = c$. [Hint: This is a similar proof to that of Problem 66. This time you need to consider a function of the form $g(x) = f(x) -$ *something*. What is *something*?] Give a counter–example to demonstrate that the claim is false if the domain of f is restricted to $(0, 1)$.

69. Prove the last part of Theorem 4.3.4, i.e. that f attains its infimum on $[a, b]$.

70. Show that if $f : [0, 1] \to \mathbb{R}$ is continuous and $0 \notin \mathrm{Ran}(f)$, then $1/f$ is bounded on $[0, 1]$ where for each $x \in [0, 1], (1/f)(x) = 1/f(x)$.

71. Explain why there are no continuous functions having domain $[0, 1]$ and range $\mathbb{R}$.

72. Suppose that the function $f : [0, 1] \to [0, 1]$ is continuous, and fix $0 < r < 1$. Suppose that we are given a sequence (x_n) in $[0, 1]$ so that $f(x_{n+1}) \leq rf(x_n)$ for all $n \in \mathbb{N}$. Show that there exists $c \in [0, 1]$ for which $f(c) = 0$. [Hint: Use the Bolzano-Weierstrass theorem.]

73. Show that for each $n \in \mathbb{N}$, the mapping $x \to x^n$ is strictly monotonic increasing on $[0, \infty)$.

74. Show that under the hypotheses of Theorem 4.3.7, the mapping f^{-1} is right continuous at a and left continuous at b.

75. Show that the function $f(x) = \sin(x)$ has a continuous inverse when restricted to the interval $[-\pi/2, \pi/2]$. What goes wrong outside this interval? [Hint: Use an appropriate trigonometric identity.]

76. Find $\displaystyle\lim_{x\to 1} \frac{1 - x}{1 - x^{m/n}}$, where $m, n \in \mathbb{N}$.

77. If $f : [a, b] \to \mathbb{R}$ is continuous and bijective, and $f(a) < f(b)$, show that f is strictly monotonic increasing.

78. * Show that if $f, g : \mathbb{R} \to \mathbb{R}$ (each having domain $\mathbb{R}$) are monotonic increasing, with $f + g$ continuous, then both f and g are continuous.

79. *

    (a) Let (x_n) be a sequence in $\mathbb{R}$ for which $x_1 = a > 0$ and $x_{n+1} = \sqrt{x_n}$, for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} x_n = 1$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function for which $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Use the result of (a) to show that f is constant.

Problems for Chapter 5

80. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in D_f$. State carefully what this means in terms of (a) limits of sequences, (b) the $\epsilon - \delta$ criterion.

81. Give a rigorous proof that $f(x) = 1/x$ is differentiable for all $x \in \mathbb{R}\backslash\{0\}$ and find $f'(x)$ explicitly. Can we extend the function so that it is differentiable on the whole of $\mathbb{R}$ by defining its value at zero to be zero?

82. Give a rigorous proof that $f(x) = e^{kx}$ is differentiable for all $x \in \mathbb{R}$, where $k \neq 0$, and find $f'(x)$ explicitly. You may use the fact that $e^{kh} - 1 - kh = g(h)$, where $\lim_{h\to 0} g(h)/h = 0$, (which follows from the series expansion).

83. Show that the function

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

is differentiable at every $x \neq 0$ but fails to be differentiable at $x = 0$.

84. Show that the function

$$f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

is differentiable at every $x \in \mathbb{R}$. What can you say about its second derivative?

85. Explain carefully at which points the function $f : \mathbb{R} \to \mathbb{R}$ for which $f(x) = x - [x]$ for all $x \in \mathbb{R}$, is differentiable, and find the value of its derivative there. (Recall that if $x \in \mathbb{R}$, $[x]$ is the integer part of x, i.e. $[x] = \max\{n \in \mathbb{Z}; n \leq x\}$.)

86 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a. Show that

$$\lim_{h\downarrow 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a).$$

By considering $f(x) = |x|$, show that this limit may exist, even when f is not differentiable at a.

87. Prove Theorem 5.2.2, i.e. show that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a if and only if it is both right and left differentiable there, and these two derivatives both agree there, in which case $f'(a)$ is their common value.

88. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0, \end{cases}$$

Determine whether

(a) f is continuous at 0.

(b) $f'(0)$ exists.

(c) f' is continuous at 0.

(d) $f''(0)$ exists.

89. Must any differentiable function $f : [a, b] \to \mathbb{R}$ have a maximum and minimum value? Why? If f is as above and $f(a) = f(b)$, must f have a maximum and minimum value in (a, b)?

90. If $a_0, a_1, \ldots, a_n$ are such that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0,$$

show that $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ has a zero in $[0, 1]$. [Hint: Integrate the function f term–by–term, and think about how to use Rollé's theorem.]

91. Use the mean value theorem to show the following:

(a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) with $f'(c) = 0$ for all $c \in (a, b)$, then f is constant on (a, b).

(b) If g and h are both continuous on $[a, b]$ and differentiable on (a, b) with $h'(x) = g'(x)$ for all $x \in (a, b)$. then there exists $k \in \mathbb{R}$ so that $h(x) = g(x) + k$, for all $x \in [a, b]$.

92. If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and there exist $m, M \in \mathbb{R}$ such that $m \leq f'(c) \leq M$, for all $c \in (a, b)$, show that

$$f(a) + m(b - a) \leq f(b) \leq f(a) + M(b - a).$$

93. Show that the restriction of $f(x) = \cos(x)$ to $[0, \pi]$ has an inverse defined on $[-1, 1]$, which is differentiable on $(-1, 1)$.

94. If $r > 0$ and $q \in \mathbb{R}$ show that the polynomial $p(x) = x^3 + rx + q$ has exactly one real zero.

95. Show that if $p > 1$ and $x \in (0, 1)$ then
$$1 - x^p < p(1 - x).$$

96. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable at a with $f'(a) = 0$. If $f''(a) < 0$, show that f has a local maximum at a, while if $f''(a) > 0$, show that f has a local minimum at a.

97. Prove the following "intermediate value theorem for derivatives": if $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $[a, b]$, f' is continuous and $\gamma \in \mathbb{R}$ is such that $f'(a) < \gamma < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = \gamma$. [Hint: Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = f(x) - \gamma x$, and show that g attains its minimum value at a point $c \in (a, b)$.]

98. Prove Cauchy's mean value theorem (Theorem 5.5.4 in the notes.) [Hint: Define $h(x) = f(x) - \rho g(x)$, where $\rho = \frac{f(b)-f(a)}{g(b)-g(a)}$.]

99. (a) Use l'Hôpital's rule to find $\lim_{x\to 0} \frac{\sin(x)}{x}$ and $\lim_{x\to 0} \frac{\cos(x)-1}{x}$.
    (b) Use the result of (a) to give a rigorous proof that $f(x) = \sin(x)$ is differentiable at every $x \in \mathbb{R}$, with $f'(x) = \cos(x)$.
    (c) Consider the function $f(x) = \sin(x)/x$ with $D_f = \mathbb{R} \setminus \{0\}$. Find a continuous extension of f to the whole of $\mathbb{R}$. Is your extended function differentiable at $x = 0$?

100. Use l'Hôpital's rule to show that if f is twice differentiable at a, then
$$f''(a) = \lim_{h\downarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

101. * The purpose of this question is to prove the following version of l'Hôpital's rule: Suppose that f and g are each differentiable on (a, b) with $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x\downarrow a} f(x) = \lim_{x\downarrow a} g(x) = \infty$, then
$$\lim_{x\downarrow a} \frac{f(x)}{g(x)} = \lim_{x\downarrow a} \frac{f'(x)}{g'(x)},$$
whenever the limit on the right hand side is finite.

(a) First show there exists $K \in (a, b)$ so that $f(x) \neq 0, g(x) \neq 0$ for all $x \in (a, K)$ and that there exists $c \in (a, K)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(K)}{g(x) - g(K)} = \frac{f(x)}{g(x)} \left(\frac{1 - f(K)/f(x)}{1 - g(K)/g(x)} \right).$$

(b) Deduce that

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \left[1 + \frac{f(K)/f(x) - g(K)/g(x)}{1 - f(K)/f(x)} \right].$$

(c) Show that $\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$.

(d) Finally, show that you can take limits as $K \to a$ to deduce the result.

102. Use Taylor (or Maclaurin's) theorem to characterise the remainder terms appearing in the following expansions (where $x \in \mathbb{R}$):

(a) $e^x = 1 + x + \cdots + \frac{x^{n-1}}{(n-1)!} + R_n(x)$.

(b) $\sin(x) = x - \frac{x^3}{3!} + \cdots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} + R_{2n}(x)$.

(c) $\cos(x) = 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n+1}(x)$.

103. Prove that for all $x \in \mathbb{R}$:

$$1 - \frac{x^2}{2} \leq \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

104. * The purpose of this question is to apply Maclaurin's theorem to prove that e is irrational. You may use the fact, proved in Example 2.6, that $2 \leq e \leq 3$. We'll seek a proof by contradiction, so assume that e is rational. Then since $e \geq 2$, we may write $e = a/b$, where $a > b$.

(a) Show that for all $n \in \mathbb{N}$,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_{n+1},$$

where $0 < R_{n+1} < \frac{3}{(n+1)!}$.

(b) Deduce that $n!e = m + n!R_{n+1}$, where $m \in \mathbb{N}$.

(c) Show that $0 < n!R_{n+1} < 1$, and then take $n > \max\{3, b\}$ to obtain the desired contradiction.