

MAS221 Analysis – Exercises II

(Problems labelled * may be more demanding.)

Problems for Chapter 3

44. Write down the domains of the following mappings from \mathbb{R} to \mathbb{R} :

$$(a) f_1(x) = \frac{x^2 + 2x + 7}{x(x + 1)},$$

$$(b) f_2(x) = \frac{(x - 1)(x + 4)}{x^3 + 4x^2 + x - 6},$$

$$(c) f_3(x) = \frac{x + 4}{x^2 + 5x + 6},$$

$$(d) f_4(x) = \exp\left\{-\frac{1}{x - 1}\right\},$$

$$(e) f_5(x) = \cos\left(\frac{1}{\pi x}\right).$$

45. Investigate $\lim_{x \rightarrow 1} f_2(x)$, $\lim_{x \rightarrow -2} f_2(x)$ and $\lim_{x \rightarrow -3} f_2(x)$, and calculate the value of the limit, when it exists.

46. If $f : \mathbb{R} \rightarrow [0, \infty)$ satisfies $\lim_{x \rightarrow a} f(x) = l$, where $l > 0$, show that $\lim_{x \rightarrow a} \sqrt{f(x)} = \sqrt{l}$. Hence calculate $\lim_{x \rightarrow 1} \sqrt{\frac{x + 1}{x^2}}$. [Hint: For the first part, use the result of Problem 25c.]

47. Why are limits of functions unique (compare with Theorem 2.1.1)?

48. Verify that $\operatorname{sgn}(x) = \frac{|x|}{x} = \frac{x}{|x|}$, for $x \neq 0$ (see Example 3.3 in the notes for the definition). Deduce that $\lim_{x \rightarrow 0} \operatorname{sgn}(x)$ does not exist. Show that both the left and right limits exist at $x = 0$, and find their values.

49. For the following functions, each of which is defined on the whole of \mathbb{R} , find every point at which both the left and right limits exist, and are different, and find the values of these limits:

$$(a) f(x) = \begin{cases} 1 - x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1, \end{cases}$$

(b) $g(x) = [x]$, where $[x]$ is the integer part of x , i.e. the greatest integer less than or equal to x .

(c) $h(x) = 3 - 5\mathbf{1}_{(0,1]}(x) + 7\mathbf{1}_{(1,2]}(x)$.

50. Consider the function $f(x) = \sin(1/x)$. What is its domain? Does it have a limit at $x = 0$? [Hint: Consider sequences whose n th term is $1/(\theta + 2n\pi)$, and think about good choices for θ .]
51. Consider the function $f(x) = x \sin(1/x)$. What is its domain? Does it have a limit at $x = 0$?
52. Prove Theorem 3.3.1 in the notes to establish $\epsilon - \delta$ criteria for left and right limits.
53. In the notes, we gave meaning to $\lim_{x \rightarrow a} f(x)$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function and $a \in \mathbb{R}$. In this question, you can investigate what happens when $a = \infty$ and $a = -\infty$.
- (a) Formulate a rigorous definition, in terms of sequences, for $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ to exist. [Hint: First decide whether you want to use *convergent* or *divergent* sequences.]
- (b) Find an analogue of the $(\epsilon - \delta)$ criterion for this case, and prove the analogous result to Theorem 3.2.2. [Hint: For the analogue of $(\epsilon - \delta)$, think very carefully about how you are going to control the behaviour of x . Remember that you cannot treat ∞ as if it were a number!]
- (c) Check that you are able to prove that $\lim_{x \rightarrow \infty} 1/x = \lim_{x \rightarrow -\infty} 1/x = 0$, using your criterion.
54. (a) Formulate a rigorous definition, in terms of sequences, for $\lim_{x \rightarrow \infty} f(x) = \pm\infty$ and $\lim_{x \rightarrow -\infty} f(x) = \pm\infty$ to exist, and write down an analogue of the $(\epsilon - \delta)$ criterion.
- (b) Let $f : \mathbb{R} \rightarrow [0, \infty)$ and $g : \mathbb{R} \rightarrow [0, \infty)$ both have domain \mathbb{R} , and suppose that $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow \infty} g(x) = l$, where $l > 0$. Define $h(x) = f(x)g(x)$ for all $x \in \mathbb{R}$. Show that $\lim_{x \rightarrow \infty} h(x) = \infty$. What happens if all instances of ∞ in the above are replaced with $-\infty$?
- (c) Let $p : \mathbb{R} \rightarrow \mathbb{R}$ be an even polynomial of degree m , where the leading coefficient (i.e. the coefficient of x^m) is positive. Show that $\lim_{x \rightarrow \infty} p(x) = \lim_{x \rightarrow -\infty} p(x) = \infty$. What happens when m is odd?

Problems for Chapter 4

55. Return to Problem 44. For each function considered there, what is the maximum subset of its domain on which it is continuous?
56. (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous at a point a .
- i. Prove that $|f|$ is continuous at a , where $|f|(x) = |f(x)|$, for all $x \in D_f$. [Hint: Use Theorem 1.3.1]
 - ii. Assuming that $f(a) \neq 0$, show that $\sqrt{|f|}$ is continuous at a , where $\sqrt{|f|}(x) = \sqrt{|f(x)|}$, for all $x \in D_f$. [Hint: Go back to Problem 46.]
- (b) Find the maximum subset of $[0, \infty)$ for which the mapping $x \rightarrow \sqrt{x}$ is continuous.
57. Prove Theorem 4.1.3 in the notes (i.e. that composition of continuous functions is continuous). [Hint: Use sequences.]
58. Define f and $g : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 1/x$ for $x \in \mathbb{R} \setminus \{0\}$, and $g(x) = 1+x^2$ for $x \in \mathbb{R}$. Write down the functions $f \circ g$, and $g \circ f$, giving their domains explicitly. What can you say about continuity of these functions?
59. For each of the functions in Problem 49,
- (a) Find the maximum subset of its domain on which it is continuous.
 - (b) Identify those discontinuities which are jumps, and calculate the size of each jump.
60. (a) Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \frac{(1+x)^2 - 1}{x}$$

- with $D_f = \mathbb{R} \setminus \{0\}$. Find an extension of f to the whole of \mathbb{R} that is continuous there.
- (b) Write down the domain D_f of the function $f(x) = \frac{x^3 - 8}{x^2 - 4}$ and explain why f is continuous at every point of D_f . The complement D_f^c of D_f in \mathbb{R} comprises two points. Show that f may be extended to be continuous at only one of these points, and write down this continuous extension.

61. Suppose that the mapping $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at a with $g(a) > 0$ (or alternatively, $g(a) < 0$). Show that there exists $\delta > 0$ so that $g(x) > 0$ (respectively, $g(x) < 0$) for all $x \in (a - \delta, a + \delta)$. [Hint: Assume $g(a) > 0$ and try a proof by contradiction.]

62. (a) For any $x, y \in \mathbb{R}$ show that

$$\max\{x, y\} = \frac{1}{2}(x + y) + \frac{1}{2}|x - y|.$$

Hence show that if both f and $g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous at a , then so is $\max\{f, g\}$, where for all $x \in D_f \cap D_g$,

$$\max\{f, g\}(x) = \max\{f(x), g(x)\}.$$

(b) Find a similar expression for $\min\{x, y\}$, and hence prove continuity of $\min\{f, g\}$ at a .

63. The aim of this question is to prove the following: the only possible functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with domain \mathbb{R} which are continuous at zero and satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ are the linear mappings $f(x) = kx$ for all $x \in \mathbb{R}$, where $k \in \mathbb{R}$ is fixed.

Begin by considering $f : \mathbb{R} \rightarrow \mathbb{R}$ with domain \mathbb{R} , such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Prove that $f(0) = 0$,

(b) Show that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$,

(c) If f is continuous at zero, prove that it is continuous at every $x \in \mathbb{R}$,

(d) If $f(1) = k$, prove that $f(n) = kn$ for all $n \in \mathbb{Z}$,

(e) If $f(1) = k$, prove that $f(p/q) = kp/q$ for all $p/q \in \mathbb{Q}$,

(f) If $f(1) = k$ and f is continuous at zero, prove that $f(x) = kx$ for all $x \in \mathbb{R}$.

64. Show that Dirichlet's "other" function, as discussed in Example 4.5 in the notes, is discontinuous at every rational point in its domain.

65. What can you say about left/right continuity of the mapping $x \rightarrow \mathbf{1}_{(a,b)}$ at the points a and b ?

66. Prove Corollary 4.3.2. [Hint: Apply the intermediate value theorem to the function $g(x) = f(x) - \gamma$, for $x \in [a, b]$.]

67. Use Corollary 4.3.2 to show that
- Every continuous function from \mathbb{R} to \mathbb{Z} is constant,
 - Every continuous function from \mathbb{R} to \mathbb{Q} is constant.
68. Prove the following *fixed point theorem*: if $f : [a, b] \rightarrow (a, b)$ is continuous, then there exists $c \in (a, b)$ such that $f(c) = c$. [Hint: This is a similar proof to that of Problem 66. This time you need to consider a function of the form $g(x) = f(x) - \text{something}$. What is *something*?] Give a counter-example to demonstrate that the claim is false if the domain of f is restricted to $(0, 1)$.
69. Prove the last part of Theorem 4.3.4, i.e. that f attains its infimum on $[a, b]$.
70. Show that if $f : [0, 1] \rightarrow \mathbb{R}$ is continuous and $0 \notin \text{Ran}(f)$, then $1/f$ is bounded on $[0, 1]$ where for each $x \in [0, 1]$, $(1/f)(x) = 1/f(x)$.
71. Explain why there are no continuous functions having domain $[0, 1]$ and range \mathbb{R} . [Hint: Use Theorem 4.3.4.]
72. Suppose that the function $f : [0, 1] \rightarrow [0, 1]$ is continuous, and fix $0 < r < 1$. Suppose that we are given a sequence (x_n) in $[0, 1]$ so that $f(x_{n+1}) \leq rf(x_n)$ for all $n \in \mathbb{N}$. Show that there exists $c \in [0, 1]$ for which $f(c) = 0$. [Hint: Use the Bolzano-Weierstrass theorem.]
73. Show that for each $n \in \mathbb{N}$, the mapping $x \rightarrow x^n$ is strictly monotonic increasing on $[0, \infty)$.
74. Show that under the hypotheses of Theorem 4.3.7, the mapping f^{-1} is right continuous at a and left continuous at b .
75. Show that the function $f(x) = \sin(x)$ has a continuous inverse when restricted to the interval $[-\pi/2, \pi/2]$. What goes wrong outside this interval? [Hint: Use an appropriate trigonometric identity.]
76. Find $\lim_{x \rightarrow 1} \frac{1-x}{1-x^{m/n}}$, where $m, n \in \mathbb{N}$.
77. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and bijective, and $f(a) < f(b)$, show that f is strictly monotonic increasing.
78. * Show that if $f, g : \mathbb{R} \rightarrow \mathbb{R}$ (each having domain \mathbb{R}) are monotonic increasing, with $f + g$ continuous, then both f and g are continuous.

79. *

- (a) Let (x_n) be a sequence in \mathbb{R} for which $x_1 = a > 0$ and $x_{n+1} = \sqrt{x_n}$, for all $n \in \mathbb{N}$. Show that $\lim_{n \rightarrow \infty} x_n = 1$.
- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function for which $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Use the result of (a) to show that f is constant.

Problems for Chapter 5

80. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $a \in D_f$. State carefully what this means in terms of (a) limits of sequences, (b) the $\epsilon - \delta$ criterion.
81. Give a rigorous proof that $f(x) = 1/x$ is differentiable for all $x \in \mathbb{R} \setminus \{0\}$ and find $f'(x)$ explicitly. Can we extend the function so that it is differentiable on the whole of \mathbb{R} by defining its value at zero to be zero?
82. Give a rigorous proof that $f(x) = e^{kx}$ is differentiable for all $x \in \mathbb{R}$, where $k \neq 0$, and find $f'(x)$ explicitly. You may use the fact that $e^{kh} - 1 - kh = g(h)$, where $\lim_{h \rightarrow 0} g(h)/h = 0$, (which follows from the series expansion).
83. Show that the function

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

is differentiable at every $x \neq 0$ but fails to be differentiable at $x = 0$.

84. Show that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

is differentiable at every $x \in \mathbb{R}$. What can you say about its second derivative?

85. Explain carefully at which points the function $f : \mathbb{R} \rightarrow \mathbb{R}$ for which $f(x) = x - [x]$ for all $x \in \mathbb{R}$, is differentiable, and find the value of its derivative there. (Recall that if $x \in \mathbb{R}$, $[x]$ is the integer part of x , i.e. $[x] = \max\{n \in \mathbb{Z}; n \leq x\}$.)

86 Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at a . Show that

$$\lim_{h \downarrow 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a).$$

By considering $f(x) = |x|$, show that this limit may exist, even when f is not differentiable at a .

87. Prove Theorem 5.2.2, i.e. show that $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at a if and only if it is both right and left differentiable there, and these two derivatives both agree there, in which case $f'(a)$ is their common value.

88. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0, \end{cases}$$

Determine whether

- (a) f is continuous at 0.
- (b) $f'(0)$ exists.
- (c) f' is continuous at 0.
- (d) $f''(0)$ exists.

89. Must any differentiable function $f : [a, b] \rightarrow \mathbb{R}$ have a maximum and minimum value? Why? If f is as above and $f(a) = f(b)$, must f have a maximum and minimum value in (a, b) ?

90. If a_0, a_1, \dots, a_n are such that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_n}{n+1} = 0,$$

show that $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ has a zero in $[0, 1]$. [Hint: Integrate the function f term-by-term, and think about how to use Rollé's theorem.]

91. Use the mean value theorem to show the following:

- (a) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) with $f'(c) = 0$ for all $c \in (a, b)$, then f is constant on (a, b) .
- (b) If g and h are both continuous on $[a, b]$ and differentiable on (a, b) with $h'(x) = g'(x)$ for all $x \in (a, b)$. then there exists $k \in \mathbb{R}$ so that $h(x) = g(x) + k$, for all $x \in [a, b]$.

92. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and there exist $m, M \in \mathbb{R}$ such that $m \leq f'(c) \leq M$, for all $c \in (a, b)$, show that

$$f(a) + m(b - a) \leq f(b) \leq f(a) + M(b - a).$$

93. Show that the restriction of $f(x) = \cos(x)$ to $[0, \pi]$ has an inverse defined on $[-1, 1]$, which is differentiable on $(-1, 1)$.
94. If $r > 0$ and $q \in \mathbb{R}$ show that the polynomial $p(x) = x^3 + rx + q$ has exactly one real zero.
95. Show that if $p > 1$ and $x \in (0, 1)$ then

$$1 - x^p < p(1 - x).$$

96. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable at a with $f'(a) = 0$. If $f''(a) < 0$, show that f has a local maximum at a , while if $f''(a) > 0$, show that f has a local minimum at a .
97. Prove the following “intermediate value theorem for derivatives”: if $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable on $[a, b]$, f' is continuous and $\gamma \in \mathbb{R}$ is such that $f'(a) < \gamma < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = \gamma$. [Hint: Consider the function $g : \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = f(x) - \gamma x$, and show that g attains its minimum value at a point $c \in (a, b)$.]
98. Prove Cauchy’s mean value theorem (Theorem 5.5.4 in the notes.) [Hint: Define $h(x) = f(x) - \rho g(x)$, where $\rho = \frac{f(b)-f(a)}{g(b)-g(a)}$.]

99. (a) Use l’Hôpital’s rule to find $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$ and $\lim_{x \rightarrow 0} \frac{\cos(x)-1}{x}$.
- (b) Use the result of (a) to give a rigorous proof that $f(x) = \sin(x)$ is differentiable at every $x \in \mathbb{R}$, with $f'(x) = \cos(x)$.
- (c) Consider the function $f(x) = \sin(x)/x$ with $D_f = \mathbb{R} \setminus \{0\}$. Find a continuous extension of f to the whole of \mathbb{R} . Is your extended function differentiable at $x = 0$?

100. Use l’Hôpital’s rule to show that if f is twice differentiable at a , then

$$f''(a) = \lim_{h \downarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

101. * The purpose of this question is to prove the following version of l'Hôpital's rule: Suppose that f and g are each differentiable on (a, b) with $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x \downarrow a} f(x) = \lim_{x \downarrow a} g(x) = \infty$, then

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \lim_{x \downarrow a} \frac{f'(x)}{g'(x)},$$

whenever the limit on the right hand side is finite.

- (a) First show there exists $K \in (a, b)$ so that $f(x) \neq 0, g(x) \neq 0$ for all $x \in (a, K)$ and that there exists $c \in (a, K)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(K)}{g(x) - g(K)} = \frac{f(x)}{g(x)} \left(\frac{1 - f(K)/f(x)}{1 - g(K)/g(x)} \right).$$

- (b) Deduce that

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \left[1 + \frac{f(K)/f(x) - g(K)/g(x)}{1 - f(K)/f(x)} \right].$$

- (c) Show that $\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$.
 (d) Finally, show that you can take limits as $K \rightarrow a$ to deduce the result.

102. Use Taylor (or Maclaurin's) theorem to characterise the remainder terms appearing in the following expansions (where $x \in \mathbb{R}$):

- (a) $e^x = 1 + x + \cdots + \frac{x^{n-1}}{(n-1)!} + R_n(x)$.
 (b) $\sin(x) = x - \frac{x^3}{3!} + \cdots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} + R_{2n}(x)$.
 (c) $\cos(x) = 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n+1}(x)$.

103. Prove that for all $x \in \mathbb{R}$:

$$1 - \frac{x^2}{2} \leq \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

104. * The purpose of this question is to apply Maclaurin's theorem to prove that e is irrational. You may use the fact, proved in Example 2.6, that $2 \leq e \leq 3$. We'll seek a proof by contradiction, so assume that e is rational. Then since $e \geq 2$, we may write $e = a/b$, where $a > b$.

(a) Show that for all $n \in \mathbb{N}$,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_{n+1},$$

where $0 < R_{n+1} < \frac{3}{(n+1)!}$.

(b) Deduce that $n!e = m + n!R_{n+1}$, where $m \in \mathbb{N}$.

(c) Show that $0 < n!R_{n+1} < 1$, and then take $n > \max\{3, b\}$ to obtain the desired contradiction.