MAS221 Analysis (Semester 1)– Solutions to Problems 1 to 7.

1. (a) $c = \frac{1}{2}(a+b)$ is rational, and satisfies $a < c < b$, for $c - a = b - c = \frac{1}{2}(b-a) > 0$.

   (b) One approach (it is not the only one) is to obtain the numbers $c_1, c_2, \ldots, c_{n-1}$ by $c_j = a + \frac{j}{n}(b-a)$ for $1 \le j \le n-1$. Clearly $c_j - a \ge 0$, while $b - c_j = \frac{n-j}{n}(b-a) > 0$. Note that in this construction, the numbers $c_1, c_2, \ldots, c_{n-1}$ are equally spaced with $c_j - c_{j-1} = (b-a)/n$ for $1 \le j \le n-1$.

   (c) Suppose there are only a finite number N (say) of distinct rational numbers lying between a and b. Then we obtain a contradiction with (b), by taking $n = N + 2$.

2. If $\sqrt{x}$ is rational, we may write $\sqrt{x} = \frac{a}{b}$ where $a \in \mathbb{Z}_+$ and $b \in \mathbb{N}$. But then $x = \frac{a^2}{b^2}$ is also rational, and that gives the desired contradiction.

   $(\sqrt{p} + \sqrt{q})^2 = p + q + 2\sqrt{pq}$. Now pq is not a perfect square, so $\sqrt{pq}$ is irrational by Theorem 1.2.3. It follows that $(\sqrt{p} + \sqrt{q})^2$ is irrational, and hence so is its square root, by the first part of this problem.

3. The answer is "yes", and here is a "non-constructive" proof. $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational. If it is rational choose $a = b = \sqrt{2}$, and if it is irrational choose $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. [In fact it can be shown that $\sqrt{2}^{\sqrt{2}}$ is irrational, but this needs some very advanced techniques.]

4. The two cases (i) and (ii) are both easy, as the inequality is an equality. For case (iii), we have

$$|a + b| = a - |b| \le |a| + |b|.$$

   For case (iv), $a \ge 0, b < 0$ and $|a| < |b|$, we have similarly

$$|a + b| = |b| - a \le |a| + |b|.$$

   The other two cases, are as in (iii) and (iv), but with the roles of a and b interchanged.

5. It works when $n = 1$ as both sides are equal. Now suppose the inequality is true for some $n \in \mathbb{N}$. Then

$$\begin{aligned} (1+x)^{n+1} &= (1+x)^n(1+x) \\ &\ge (1+nx)(1+x) \\ &= 1 + (n+1)x + nx^2 \\ &\ge 1 + (n+1)x, \end{aligned}$$

and the result is proved by induction. When $x > 0$, the binomial theorem yields

$$(1+x)^n = 1+nx+\frac{1}{2}n(n-1)x^2+\frac{1}{6}n(n-1)(n-2)x^3+\cdots+x^n \geq 1+nx$$

6. The simplest route is to expand

$$\begin{aligned} a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right] &= a\left[x^2+\frac{bx}{a}+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right] \\ &= ax^2+bx+c. \end{aligned}$$

If $b^2 \leq 4ac$ then it is clear that $f(x) \geq 0$. Conversely if $f(x) \geq 0$ for all x, then in particular $f\left(-\frac{b}{2a}\right) \geq 0$ and so $4ac - b^2 \geq 0$ as required.

7.

$$\begin{aligned} f(x) &= \sum_{i=1}^{n}(a_i x + b_i)^2 \\ &= \left(\sum_{i=1}^{n} a_i^2\right)x^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right)x + \sum_{i=1}^{n} b_i^2. \end{aligned}$$

The result now follows by applying the result of Problem 6 where we take a, b and c (respectively) to be $a = \sum_{i=1}^{n} a_i^2, b = 2\sum_{i=1}^{b} a_i b_i$ and $c = \sum_{i=1}^{n} b_i^2$.