MAS221 Analysis (Semester 1)– Solutions to Problems 66 to 82

66. Since f is continuous on $[a, b]$, so is g. We have $g(a) = f(a) - \gamma < 0$ and $g(b) = f(b) - \gamma > 0$. Hence by the intermediate value theorem (Theorem 4.3.1), there exists $c \in (a, b)$ so that $g(c) = 0$, i.e. $f(c) = \gamma$, as was required.

67. (a) If f is continuous on $\mathbb{R}$ it is continuous on $[a, b]$ for each $a < b$. If f is not a constant, we must be able to find a, b such that $f(a) \neq f(b)$. Now either $f(a) < f(b)$ or $f(a) > f(b)$. Assume the former (without loss of generality). Then there exists $m, n \in \mathbb{Z}$ with $m < n$ such that $f(a) = m$ and $f(b) = n$. Hence by Corollary 4.3.2, there exists $c \in (a, b)$ so that $f(c) = m + 1/2$, and that is the desired contradiction.

    (b) Argue as in (a), using the fact that between any two rational numbers, we can find an irrational number (recall Theorem 1.2.2).

68. *Something* is x. So apply the intermediate value theorem to $g(x) = f(x) - x$. Since $\text{Ran}(f) \subseteq (a, b)$ we have $f(a) > a$ and $f(b) < b$, and so $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. Hence there exists $c \in (a, b)$ such that $g(c) = 0$, i.e. $f(c) = c$. For the counter–example, consider $f(x) = 1/x$. It is continuous on $(0, 1)$ but there is no $c \in (0, 1)$ for which $1/c = c$.

69. Define $\gamma = \inf_{x \in [a,b]} f(x)$ and assume that it is not attained, and so $\gamma < f(x)$ for all $x \in [a, b]$. Then $h(x) = \dfrac{1}{f(x) - \gamma}$ is continuous, and hence bounded on $[a, b]$. So there exists $L \geq 0$ so that $|h(x)| \leq L$ for all $x \in [a, b]$. By Problem 15 b), given any $\epsilon > 0$, there exists $x \in [a, b]$ such that $f(x) < \gamma + \epsilon$. Now take $\epsilon = 1/K$ to deduce that $h(x) > K$, which yields the required contradiction.

70. By algebra of limits, $1/f$ is continuous on $[0, 1]$ and so is bounded by Theorem 4.3.4.

71. If f is continuous on $[0, 1]$, then it is bounded by Theorem 4.3.4, and so there exists $L \geq 0$ so that $|f(x)| \leq L$ for all $x \in [0, 1]$. Hence $\text{Ran}(f) \subseteq [-L, L]$.

72. Since (x_n) is bounded, by the Bolzano–Weierstrass theorem (Theorem 2.4.3) it has a convergent subsequence (x_{n_k}). Let $\lim_{k \to \infty} x_{n_k} = c$ and note that $c \in [0, 1]$ by Problem 31. By induction we have $f(x_{n_k}) \leq$

$r^{n_k-1}f(x_{n_1})$ and so $0 \le f(x_{n_k}) \le r^{n_k-1}$, since $\mathrm{Ran}(f) \subseteq [0,1]$. Now use continuity of f and the sandwich rule to deduce that:

$$f(c) = \lim_{k\to\infty} f(x_{n_k}) = 0,$$

since $r < 1$.

73. For all $x > y$,

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-1} + y^{n-1}) > 0.$$

74. We'll just prove right continuity of f^{-1} at $f(a)$. The argument for left continuity at $f(b)$ is very similar. We use the same notation as in the proof of Theorem 4.3.7. We must show that given any $\epsilon > 0$, there exists $\delta > 0$ so that if $y - f(a) < \delta$ then $|f^{-1}(y) - a| < \epsilon$. Given ϵ, choose $x_1 > a + \epsilon$. Since f is increasing, we have $f(x_1) > f(a)$. Now if $f(a) < y < f(x_1)$ then $f^{-1}(y) - a = f^{-1}(y) - f^{-1}(f(a)) = |f^{-1}(y) - f^{-1}(f(a))| < \epsilon$. Now let $\delta = f(x_1) - f(a)$ and the result follows.

75. For all $-\pi/2 \le x < y \le \pi/2$,

$$\sin(y) - \sin(x) = 2\sin\left(\frac{y-x}{2}\right)\cos\left(\frac{y+x}{2}\right) > 0,$$

so the function is strictly monotonic increasing on this interval. It is also continuous (stated in notes), and so by Theorem 4.3.7, it is has a continuous inverse $f^{-1}(x) = \arcsin(x)$ (or $\sin^{-1}(x)$) defined on $[-1,1]$. Outside the interval $[-\pi/2, \pi/2]$, the function f might fail to be strictly increasing. In fact it is strictly decreasing on each interval of the form $((4n+1)\pi/2, (4n+3)\pi/2)$, and strictly increasing on each interval of the form $((4n-1)\pi/2, (4n+1)\pi/2)$, where $n \in \mathbb{Z}$.

76. Let $y = x^{1/n}$. Then if $f : \mathbb{R} \to \mathbb{R}$ is continuous in a neighbourhood of 1, $\lim_{y\to 1} f(y) = \lim_{x\to 1} f(x^{1/n})$, as the mapping $x \to x^{1/n}$ is continuous (see Example 4.5). Then

$$\frac{1-x}{1-x^{m/n}} = \frac{1-y^n}{1-y^m}.$$

Now for $x \ne 1, y \ne 1$ and taking $f(y) = \frac{1-y^n}{1-y^m}$, we have

$$\frac{1-y^n}{1-y^m} = \frac{y^{n-1} + y^{n-2} + \cdots + y + 1}{y^{m-1} + y^{m-2} + \cdots + y + 1},$$

and so $\lim_{y\to 1} \dfrac{1-y^n}{1-y^m} = n/m$.

77. If $a < x < b$, we have $f(a) < f(x) < f(b)$. To see this note that if $f(a) \geq f(x)$ then either $f(a) = f(x)$, or $f(a) > f(x)$. The first of these cannot hold as f is one-to-one and hence injective. For the second, we apply Corollary 4.3.2 on the interval $[x, b]$ to deduce that there exists $c \in (x, b)$ such that $f(c) = f(a)$, and this again violates the injectivity of the mapping f. A similar argument can be used to show that we cannot have $f(b) \leq f(x)$. Applying this result to the interval $[x, b]$ we see that if $a < x < y < b$, then $f(a) < f(x) < f(y) < f(b)$ and so f is strictly monotonic increasing.

78. First observe that $f+g$ is monotonic increasing since both f and g are. Choose $a \in \mathbb{R}$. Given $\epsilon > 0$, there exists $\delta > 0$ so that if $x > a+\delta$ then

$$|f(x) + g(x) - f(a) - g(a)| = f(x) + g(x) - f(a) - g(a) < \epsilon,$$

and so

$$|f(x) - f(a)| = f(x) - f(a) < \epsilon + g(a) - g(x) < \epsilon,$$

as g is increasing. This proves that g is right–continuous at a. A similar argument (interchanging the roles of a and x) proves that it is left–continuous, and hence continuous at a. Then $g = (f+g) - f$ is the difference of two continuous functions, and hence is itself continuous.

79. (a) By induction, for all $n \in \mathbb{N}, x_n = a^{\frac{1}{2^{n-1}}} \to 1$, as $n \to \infty$, by Proposition 2.4.1 and Problem 33(b), as (x_n) is a subsequence of $(a^{1/n})$.

    (b) For all $a \in \mathbb{R}, n \in \mathbb{N}, f(a) = f(a^{1/2}) = \cdots = f(a^{\frac{1}{2^{n-1}}})$, and so by continuity,

$$f(a) = \lim_{n\to\infty} f(a^{\frac{1}{2^{n-1}}}) = f\left(\lim_{n\to\infty} a^{\frac{1}{2^{n-1}}}\right) = f(1).$$

80. (a) There exists $f'(a) \in \mathbb{R}$, so that given any sequence (h_n) that converges to 0, we have $\displaystyle\lim_{n\to\infty} \frac{f(a+h_n) - f(a)}{h_n} = f'(a)$.

    (b) Given $\epsilon > 0$, there exists $\delta > 0$ so that if $0 < |h| < \delta$, then $\displaystyle\left|\frac{f(a+h) - f(a)}{h} - f'(a)\right| < \epsilon$.

81.

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) \\ &= -\frac{1}{x(x+h)} \to -\frac{1}{x^2}, \text{ as } h \to 0. \end{aligned}$$

We cannot differentiate at zero, as the extended function fails to be continuous there since $\lim_{x\downarrow 0}\frac{1}{x}=\infty$.

82.

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\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}(e^{k(x+h)}-e^{kx}) \\
&= e^{kx}\frac{1}{h}(e^{kh}-1) = e^{kx}\left(k+\frac{g(h)}{h}\right) \to ke^{kx}, \text{ as } h \to 0.
\end{aligned}
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