MAS221 Analysis (Semester 1)– Solutions to Problems 85 to 104 (Excluding Homework 5 Problems)

85. f is differentiable for all $x \in \mathbb{R}\setminus\mathbb{Z}$. For such points, taking h sufficiently small, we have $\frac{(x+h)-[x+h]-x+[x]}{h} = \frac{h}{h}$, and so $f'(x) = 1$. If $x \in \mathbb{Z}$, then f is not continuous and so cannot be differentiable (see Problem 49(b)).

86. First observe that if f is differentiable at a, then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}.$$

So

$$\begin{aligned} \frac{f(a+h) - f(a-h)}{2h} &= \frac{1}{2}\left(\frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h}\right) \\ &\to \frac{1}{2}2f'(a) = f'(a), \text{ as } h \to 0. \end{aligned}$$

87. Define $g_a : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ by

$$g_a(h) = \frac{f(a+h) - f(a)}{h} \text{ if } h \neq 0.$$

First suppose that f is differentiable at a, then $f'_+(a) = \lim_{h\downarrow 0} g_a(h) = f'(a)$ and $f'_-(a) = \lim_{h\uparrow 0} g_a(h) = f'(a)$. Conversely, suppose that $f'_+(a) = f'_+(a) = l$ (say). Then $\lim_{h\downarrow 0} g_a(h) = \lim_{h\uparrow 0} g_a(h)$ and so $\lim_{h\to 0} g_a(h)$ exists by Theorem 3.3.2, and equals l. But then f is differentiable and $l = f'(a)$.

89. f is differentiable on $[a, b]$ and hence continuous on $[a, b]$ by Theorem 5.2.1, so it attains its sup and inf on $[a, b]$ by Theorem 4.3.4, and these are the maximum and minimum (respectively). If $f(a)$ is not the maximum (or minimum) value, then this must occur in (a, b). If $f(a) = f(b)$ is both the maximum and minimum value, then f is constant, and the value occurs in (a, b).

90. Define $g : [0, 1] \to \mathbb{R}$ by $g(x) = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_n}{n+1}x^n$, then g is differentiable on $[0, 1]$, with $g(0) = g(1) = 0$. So by Rollé's theorem, there exists $c \in (0, 1)$ such that $g'(c) = 0$, i.e.$f'(c) = 0$.

92. By the mean value theorem: $f(b) = f(a) + f'(c)(b - a)$ and so, since f' is bounded above and below on (a, b) we have

$$f(a) + \inf_{c\in(a,b)} f'(c)(b - a) \leq f(b) \leq f(a) + \sup_{c\in(a,b)} f'(c)(b - a),$$

and the result follows.

93. f is differentiable on $[0, \pi]$ with $f'(x) = -\sin(x) < 0$ on $(0, \pi)$. So by Corollary 5.5.2 it is strictly decreasing on $[0, \pi]$. Hence by Theorem 4.3.7 it has a strictly decreasing inverse on $[-1, 1]$ which is continuous on $(-1, 1)$. This is precisely the function $f^{-1}(x) = \cos^{-1}(x)$ or $\arccos(x)$. But $x \to -\sin(x)$ is differentiable, hence continuous on $(0, \pi)$, so by Theorem 5.5.3, f^{-1} is differentiable on $(-1, 1)$, and standard calculus yields $(f^{-1})'(x) = -\frac{1}{\sqrt{1-x^2}}$ for all $x \in (-1, 1)$.

94. The polynomial p is of odd degree so it has at least one real root by Corollary 4.3.3. Also $p'(x) = 3x^2 + r > 0$, for all $x \in \mathbb{R}$, so p is strictly monotonic increasing on any closed interval $[a, b]$, and hence on the whole of $\mathbb{R}$ by Corollary 5.5.2. Then by Theorem 4.3.7, p is invertible and hence injective, and the result follows.

95. Apply the mean value theorem to the function $f(x) = x^p$ on the interval $[x, 1]$. Then there exists $c \in (x, 1)$ such that

$$1 - x^p = pc^{p-1}(1 - x) < p(1 - x).$$

96. Consider the first case. Here we have $f''(a) = \lim_{h\to 0} \frac{f'(a+h)-f'(a)}{h} = \lim_{h\to 0} \frac{f'(a+h)}{h} < 0$. In particular, $f''_+(a) = \lim_{h\downarrow 0} \frac{f'(a+h)}{h} < 0$, and so $f'(a + h) < 0$ for sufficiently small h, (say $0 < h < \delta$), in which case, by Corollary 5.5.2, f is strictly decreasing on $[a, a + \delta]$. We also have $f''_-(a) = \lim_{h\uparrow 0} \frac{f'(a+h)}{h} < 0$, and so $f'(a + h) > 0$ for sufficiently small h, (say $-\delta_1 < h < 0$), in which case, by Corollary 5.5.2, f is strictly increasing on $[a - \delta_1, a]$. Then it follows that f has a local minimum at a. The other case is similar.

97. g is differentiable and hence continuous on $[a, b]$. We have $g'(b) = f'(b) - \gamma > 0$ and $g'(a) = f'(a) - \gamma < 0$, so a and b are not extreme points for g. But g must attain its sup and inf in $[a, b]$ by Theorem 4.3.4. Since f' is continuous on $[a, b]$, so is g', and by Problem 61, $g' > 0$ in a neighbourhood of b and $g' < 0$ in a neighbourhood of a. Then by Corollary 5.5.2, g is increasing in a neighbourhood of b and decreasing in a neighbourhood of a. So it attains its minimum in (a, b), and this is an extreme point. Hence $g'(c) = 0$ and the result follows.

99. (a) $\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \cos(x) = 1, \lim_{x\to 0} \frac{\cos(x)-1}{x} = \lim_{x\to 0} \sin(x) = 0$.

    (b) Using a standard trig. identity,

$$\frac{\sin(x + h) - \sin(x)}{h} - \cos(x)$$

$$
\begin{aligned}
&= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} - \cos(x) \\
&= \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h} - 1\right) \\
&\to 0 \text{ as } h \to 0,
\end{aligned}
$$

where we have used (a).

(c) The required continuous extension is given by:

$$\tilde{f}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

It is differentiable at zero, as for all $h \neq 0$,

$$\frac{\tilde{f}(h) - \tilde{f}(0)}{h} = \frac{\sin(h)}{h^2} - \frac{1}{h}.$$

Now by using Maclaurin's theorem, we have $\sin(h) = h - \frac{h^3}{6}\cos(\theta h)$, where $0 < \theta < 1$. So
$\frac{\tilde{f}(h)-\tilde{f}(0)}{h} = -\frac{h}{6}\cos(\theta h) \to 0$ as $h \to 0$.

100. Applying l'Hôpital's rule to the given expression yields

$$\lim_{h \downarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \lim_{h \downarrow 0} \frac{f'(a+h) - f'(a-h)}{2h},$$

and then the result follows from the result of Problem 86.

101. (a) Since the left limits diverge to infinity at a it follows that given any $R > 0$, there exists $K \in (a, b)$ so that $f(x) > R$ and $g(x) > R$ for all $a < x < K$. Just choose $R = 1$ (say) and we are done with the first part. The second part is an application of Cauchy's mean value theorem, followed by a little algebra.

(b) Multiply both sides of the result of (a) by $\frac{1-g(K)/g(x)}{1-f(K)/f(x)}$. The rest is algebra.

(c) Just take limits as $x \downarrow a$ of both sides in (b).

(d) We have $c = K + \theta(K - a)$, where $0 \leq \theta \leq 1$. Then

$$0 \leq |c - a| = |K - a|.|1 + \theta| \leq 2|K - a|,$$

so by the sandwich rule, $c \to a$ as $K \to a$, and the result follows by taking limits as $K \to a$ on both sides of the result of (c).

102. (a) $\frac{x^n}{n!}e^{\theta x}$, (b) $(-1)^n \frac{x^{2n}}{(2n)!}\sin(\theta x)$, (c) $(-1)^{n+1}\frac{x^{2n}}{(2n)!}\sin(\theta x)$, where in all three cases $0 < \theta < 1$.

103. Using Maclaurin's theorem, we can find $0 < \theta < 1$ and $0 < \phi < 1$, so that for all $x \in \mathbb{R}$,

$\cos(x) = 1 - \frac{x^2}{2}\cos(\theta x) \geq 1 - \frac{x^2}{2}$, since $\cos(\theta x) \leq 1$, and

$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}\cos(\phi x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$, since $\cos(\phi x) \leq 1$.

104 (a) Applying Maclaurin's theorem, as in Problem 102(a), and putting $x = 1$, yields:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_{n+1},$$

where $R_{n+1} = \frac{e^\theta}{(n+1)!}$ and $0 < \theta < 1$. Clearly $R_{n+1} > 0$. Since $x \to e^x$ is monotonic increasing, we have $e^\theta \leq e \leq 3$, and the result follows.

(b) We have

$$\frac{n!a}{b} = n! + n! + \frac{n!}{2!} + \cdots + \frac{n!}{(n-1)!} + 1 + n!R_{n+1},$$

and clearly $m = 2n! + \frac{n!}{2!} + \cdots + \frac{n!}{(n-1)!} + 1$ is a natural number.

(c) By (a) $0 < n!R_{n+1} < 3/n + 1 < 3/4 < 1$, since $n > 3$; but since $n > b$, we have $\frac{n!a}{b} \in \mathbb{N}$, and so by (b), $n!R_{n+1} = \frac{n!a}{b} - m \in \mathbb{N}$, and this gives our required contradiction.