MAS221 Analysis (Semester 1)– Solutions to Homework 5 Problems

83. Using the product and chain rules for differentiation, if $x \neq 0$, $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$, but $\frac{f(x)-f(0)}{x} = \sin(1/x)$ has no limit as $x \to 0$ (see Problem 50).

84. For $x \neq 0, f'(x) = 2x\sin(1/x) - \cos(1/x)$. At $x = 0$,

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0} x\sin(1/x) = 0,$$

by Problem 51. So $f'(0) = 0$. For the second derivative, with $x \neq 0$,

$$f''(x) = 2\sin\left(\frac{1}{x}\right) - \frac{2}{x}\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\sin\left(\frac{1}{x}\right),$$

but f'' doesn't exist at $x = 0$, by Problem 83.

88. (a) f is continuous at zero as $f(0) = \lim_{x\uparrow 0} f(x) = \lim_{x\downarrow 0} f(x)$.

(b) $f'(0)$ exists and is zero. To see this compute $f'_+(0) = \lim_{h\downarrow 0}\frac{h^2}{h} = 0 = \lim_{h\uparrow 0}\frac{-h^2}{h} = f'_-(0)$.

(c) f' is continuous at zero since

$$f'(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases},$$

so $f'(0) = \lim_{x\uparrow 0} f'(x) = \lim_{x\downarrow 0} f'(x) = 0$.

(d) $f''_+(0) = \lim_{h\downarrow 0}\frac{2h-0}{h} = 2, f''_+(0) = \lim_{h\uparrow 0}\frac{-2h-0}{h} = -2$, and so $f''(0)$ does not exist.

91. (a) Apply the mean value theorem to any interval $[x, y]$ with $a \leq x < y \leq b$, to find there exists $c \in (x, y)$ for which

$$f(y) - f(x) = f'(c)(y - x) = 0.$$

(b) Define $f = h - g$, then the function f satisfies all the conditions of (a), and the result is immediate.

98. The function h from the hint is continuous on $[a, b]$ and differentiable on (a, b). After some algebraic manipulation, we obtain

$$h(a) = h(b) = \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)},$$

and so by Rollé's theorem, there exists $c \in (a, b)$ so that $h'(c) = 0$ and so $f'(c) = \rho g'(c)$, and the result follows.