MAS221 Analysis (Semester 1)– Solutions to Problems 8 to 19.

8. (a) $|x| < c$ if and only if $\max\{x, -x\} < c$, and this holds if and only if both $x < c$ and $-x < c$, or equivalently $x < c$ and $x > -c$, and the last two inequalities can be combined into one: $-c < x < c$.

   (b) Replace x in (a) by $x - l$, and replace c by ϵ. Then the result of (a) tells us that $|x - l| < \epsilon$ if and only if $-\epsilon < x - l < \epsilon$, and the result follows when we add l to every term within the last inequality.

9. (i) (a) (e) and (i) are the only finite sets (note that (i) is the empty set $\emptyset$, which contains no elements).

   (ii) The maximum elements for the finite sets (a) and (e) are 8 and 99, respectively. The following infinite sets have maximum elements: (b) 66, (d) 2000, (h) 3.

   (iii) The minimum elements for the finite sets (a) and (e) are 2 and 3, respectively. The following infinite sets have minimum elements: (d) -2000, (h) 0.

10. (i) They are all bounded above except for (c). Upper bounds are e.g. (a) 8, (b) 66, (d) 2000, (e) 99, (f) 6, (g) 8, (h) 3, (i) 0 (any real number will do here).

    (ii) They are all bounded below except for (c) and (f). Upper bounds are e.g. (a) 2, (b) 10, (d) -2000, (e) 3, (g) 2, (h) 0, (i) -12 (any real number will again do here).

    (iii) They are all bounded except for (c) and (f). Values of M are e.g. (a) 8, (b) 66, (d) 2000, (e) 99, (g) 8, (h) 3, (i) -666.

11. (a) is false, e.g. consider $[0, 1]$ which is infinite, but has maximum 1 and minimum 0. (b) is true. If it has a minimum c, it is bounded below and c is the infimum. (c) is false, e.g. $(0, 1]$ has infimum 0. (d) is false, e.g. the inf and sup of $\{1\}$ are both 1. (e) is false, e.g the set $\{1, 1.4, 1.41, 1.414, 1.4141, 1.41412, \ldots\}$ of rational approximations to $\sqrt{2}$ is bounded above (e.g. by 2, which is rational), but its supremum is $\sqrt{2}$. (f) is true. The empty set $\emptyset$ has this property (but it is the only set that does).

12. Suppose $|x| \leq K$ for all $x \in A$. Then by Problem 8, $-K \leq x \leq K$ for all $x \in A$ and so A is bounded above (by K), and below (by $(-K)$ and so it is bounded. Conversely suppose that A is bounded. Then it is bounded above and below and so there exist $L, M \in \mathbb{R}$ so that $L \leq x \leq M$ for all $x \in A$. Then choose $K = \max\{|L|, |M|\}$, and the

result follows since we then have $-K \leq L \leq x \leq M \leq K$ for all $x \in A$. Note that we can always ensure that $K > 0$ by choosing $|L| \neq |M|$, so taking $L = M = 0$ is ruled out. In any case, this last possibility can only occur when $A = \{0\}$.

13. If M is an upper bound for A then αM is an upper bound for αA, since for all $x \in A$, $x \leq M$, implies that $\alpha x \leq \alpha M$. We claim that $\sup(\alpha A) = \alpha \sup(A)$. For if β is a smaller upper bound for αA, we can write $\beta = \alpha\gamma$, where $\gamma = \beta/\alpha$. Then for all $x \in A, \alpha x \leq \alpha\gamma < \alpha \sup(A)$, implies that for all $x \in A, x \leq \gamma < \sup(A)$. Then γ is a smaller upper bound for A than $\sup(A)$, which yields our desired contradiction.

    If $\alpha < 0$, write $\alpha = -\alpha'$ where $\alpha' > 0$. Then $\alpha A = -\alpha' A$. We have just proved that $\alpha' A$ is bounded above, so $\alpha A = -\alpha' A$ is bounded below, and if we now assume that the set A is bounded, then

    $$\sup(\alpha A) = \sup(-\alpha' A) = -\inf(\alpha' A) = -\alpha' \inf(A) = \alpha \inf(A).$$

    The fact that $\inf(\alpha' A) = \alpha' \inf(A)$ is by a completely analogous argument to the one we've given for sup.

14. If L_1 is a lower bound for A and L_2 is a lower bound for B, then $\min\{L_1, L_2\}$ is a lower bound for $A \cup B$.

    Write $\alpha = \min\{\inf(A), \inf(B)\}$. Let $x \in A \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \geq \inf(A) \geq \alpha$, while if $x \in B$, then $x \geq \inf(B) \geq \alpha$. This proves that α is a lower bound for $A \cup B$.

    To show it is the greatest of these, suppose that $\beta > \alpha$ is a lower bound for $A \cup B$. Since either $\alpha = \inf(A)$ or $\alpha = \inf(B)$, we may suppose (without loss of generality) that $\alpha = \inf(B)$. Then for all $x \in B, x \geq \beta > \inf(B)$, so β is a greater lower bound for B than its inf, and this gives us our required contradiction.

15. (a) Suppose that $s' < s$ is an upper bound for A. Then $s' = s - \eta$, for some $\eta > 0$, and we have that for all $\epsilon > 0$, there exists $a \in A$ so that

    $$s - \epsilon < a \leq s - \eta.$$

    Just take $\epsilon = \eta$ to get a contradiction.

    (b) First the statement: "If A is non-empty and bounded below, given any $\epsilon > 0$, there exists $a \in A$ so that $a < \inf(A) + \epsilon$."

    To prove this, just observe that if it is false, then then we can find $\epsilon > 0$ such that $a \geq \inf(A) + \epsilon$ for all $a \in A$, but then $\inf(A) + \epsilon$ is a greater lower bound for A than $\inf(A)$, and that is our desired contradiction.

The second statement is "If $A \subset \mathbb{R}$ is non-empty and bounded below, and t is a lower bound for A which is such that for all $\epsilon > 0$, there exists $a \in A$ so that $a < t + \epsilon$, then $t = \inf(A)$."

To prove this, suppose that $t' > t$ is a lower bound for A. Then $t' = t + \eta$, for some $\eta > 0$, and we have that for all $\epsilon > 0$, there exists $a \in A$ so that

$$t + \eta \leq a < t - \epsilon$$

Once again, take $\epsilon = \eta$ to get a contradiction.

16. (a) is false. Consider the empty set, but it would be true if "set" were changed to "non-empty" in the question.
    (b) is false, for if L is a lower bound, then so is $L - 1$.
    (c) is true by the completeness property, for if the set has an infimum, then it is non-empty.
    (d) is true. Just take the set to be a singleton, e.g. $\{0\}$.
    (e) is true. Take $E = (0, \pi]$.
    (f) is false, e.g. consider $\{1, 3\}$.

17. (a) is true. $\sup(F)$ is an upper bound for E and so it cannot be smaller than $\sup(E)$.
    (b) is false, e.g. $E = (1, 2), F = (0, 2)$ both have supremum 2.
    (c) is true. Since $\sup(E)$ is not an upper bound for F, then there exists $y \in F$ so that $y > \sup(E)$.
    (d) is false. Take $E = F = (0, 1)$.
    (e) is false. Take $E = \{0\}$ and $F = (-1, 0)$. Then $\sup(E) = \sup(F) = 0$, but $\sup(E - F) = 1$.

18. Write $A = \left\{a - 1, a - \frac{1}{2}, a - \frac{1}{3}, \ldots, a - \frac{1}{n}, \ldots\right\}$. From this it is clear that A is bounded below, as it has a minimum $a - 1$, so $\inf(A) = a - 1$. We claim that it is bounded above (and so is bounded). Clearly a is an upper bound for if $a - \frac{1}{n} > a$ for some n, then $\frac{1}{n} < 0$, which is impossible. In fact $a = \sup(A)$. To see this, suppose that α is a smaller upper bound for A. Then we have $a - \frac{1}{n} \leq \alpha < a$ for all $n \in \mathbb{N}$. By

the Archimedean property of the real numbers, given any $\epsilon > 0$ we can find a natural number n_0, so that $1/n_0 < \epsilon$ and so

$$a - \epsilon < a - \frac{1}{n_0} \leq \alpha < a.$$

This gives the contradiction we needed (since α must be of the form $a - \epsilon$ for some $\epsilon > 0$), and so we conclude that $\sup(A) = a$.

19. Since for all $n \in \mathbb{N}$, $a_n \leq \sup_{n\in\mathbb{N}} a_n$ and $b_n \leq \sup_{n\in\mathbb{N}} b_n$, when we add these inequalities, we get that for all $n \in \mathbb{N}$,

$$a_n + b_n \leq \sup_{n\in\mathbb{N}} a_n + \sup_{n\in\mathbb{N}} b_n.$$

Then $\sup_{n\in\mathbb{N}} a_n + \sup_{n\in\mathbb{N}} b_n$ is an upper bound for the sequence $(a_n + b_n, n \in \mathbb{N})$, and $\sup_{n\in\mathbb{N}}(a_n + b_n)$ cannot exceed this value. For an example of strict inequality take $a_n = 1 - 1/n$ and $b_n = 1/n$. Then $\sup_{n\in\mathbb{N}}(a_n + b_n) = 1$, but $\sup_{n\in\mathbb{N}} a_n + \sup_{n\in\mathbb{N}} b_n = 1 + 1 = 2$.

If you take the inequality just proved, replace a_n and b_n by $-a_n$ and $-b_n$ (respectively), and then multiply both sides of the inequality by -1, then you obtain

$$\inf_{n\in\mathbb{N}}(a_n + b_n) \geq \inf_{n\in\mathbb{N}}(a_n) + \inf_{n\in\mathbb{N}}(b_n).$$

Alternatively you can prove this by a similar argument to the one used for sup. For an example of strict inequality take $a_n = 1 + 1/n$ and $b_n = 1 - 1/n$. Then $\inf_{n\in\mathbb{N}}(a_n + b_n) = 2$, but $\inf_{n\in\mathbb{N}} a_n + \inf_{n\in\mathbb{N}} b_n = 1 + 0 = 1$.