MAS221 Analysis (Semester 1)– Solutions to Problems 20 to 24.

20. Limit is 1. (a) (i) 30, (ii) 300, (iii) 3000, (iv) 30000, (v) 30000000000.

    (b) Take $N \in \mathbb{N}$ to be the smallest natural number so that $N+1$ exceeds $\frac{3}{\epsilon}$. The existence of such an N is guaranteed by the Archimedean property of $\mathbb{R}$.

    Let's just check we fully understand where N came from. To satisfy the definition of convergence, given any $\epsilon > 0$, we need to find $N \in \mathbb{N}$ so that $|1 + 3/n - 1| < \epsilon$ for all $n > N$, i.e. $3/n < \epsilon$ for all $n > N$, i.e. $n > \epsilon/3$ for all $n > N$. So we need to find N so that each of $N + 1, N + 2, N + 3, ...$ are greater than $3/\epsilon$, and that's precisely what we've done.

21. Limits are (a) 1, (b) 0, (c) 0, (d) 0. Take N to be the smallest natural number so that $N+1$ exceeds (a) $\frac{1}{\epsilon}$, (b) $\frac{3}{\epsilon}$, (c) $\frac{1}{\sqrt{\epsilon}}$, (d) $\frac{1}{\epsilon^2}$. In each case, your approach should be as in the solution to Problem 20.

22. nth term is $\frac{n}{2n-1}$. Proceed as in Problem 20. Given $\epsilon > 0$, use the Archimedean property to find $N \in \mathbb{N}$ so that $N > \frac{1+2\epsilon}{4\epsilon}$. Then straightforward algebra shows that for $n > N$ we get $\left|\frac{n}{2n-1} - \frac{1}{2}\right| < \epsilon$, as required.

23. Since (x_n) converges to x, given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that if $n > N$ then $|x_n - x| < \epsilon$. But then by Theorem 1.3.1,

    $$||x_n - |x|| \leq |x_n - x| < \epsilon,$$

    and the result follows. Converse is false in general, - e.g consider $(-1)^n$. However, the converse is true in the case where (x_n) is a null sequence (see Problem 30), i.e. $x = 0$. For in that case, for all $n \in \mathbb{N}$,

    $$||x_n| - 0| = |x_n| = |x_n - 0|.$$

24. Given any $\epsilon > 0$ we can find $N \in \mathbb{N}$ so that if $n > N$ then

    $$|b_n - 0| = b_n \leq a_n = |a_n - 0| < \epsilon.$$