MAS221 Analysis (Semester 1)– Solutions to Problems 25 to 34.

25. (a)
$$(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab} \geq a+b = (\sqrt{a+b})^2.$$
The result follows from the fact that if $x, y \geq 0$, then $y \geq x$ if and only if $\sqrt{y} \geq \sqrt{x}$ (you may want to think about why this is true).

(b) Using the triangle inequality and (a),
$$\sqrt{|a|} \leq \sqrt{|a-b|+|b|} \leq \sqrt{|a-b|}+\sqrt{|b|}.$$
Now proceed as in the proof of Theorem 1.3.1.

(c) Given $\epsilon > 0$, there exists $N \in \mathbb{N}$, so that for all $n > N$, $|a_n - l| < \epsilon^2$. Now by (b),
$$|\sqrt{|a_n|}-\sqrt{|l|}| \leq \sqrt{|a_n - l|} < \sqrt{\epsilon^2} = \epsilon.$$

26. Limits are a) 6, b) 1, c) $\frac{2}{5}$, d) $\frac{1}{2}$, e) 0.

Hint for (e) - multiply top and bottom by $\sqrt{n+1}+\sqrt{n}$.

27. (a) "For some" should be "For all."

(b) As (a) and no connection is given between ϵ and N.

(c) Some understanding shown here but there is insufficient mathematical precision.

(d) "there is some $n > N$" should read "there exists N such that for all $n > N$".

(e) As in (a) but also N appears to be defined with respect to a fixed reference n and the last phrase in imprecise.

28. (a) Instead of ϵ being fixed and then N depending on this choice we have to find an N that works for all values of ϵ. So ridiculous convergence is going to be harder to achieve than usual convergence, as (b) demonstrates.

(b) We must show that $\frac{1}{n} < \epsilon$ for all $n > N$. But this means that $n > \frac{1}{\epsilon}$ for such n. Suppose that such an N can be found and choose $\epsilon = \frac{1}{N+2}$. Now take $n = N+1$. What do you find?

29. Begin by rewriting the definition of a limit using η instead of ϵ. So (x_n) converges to x if given any $\eta > 0$ there exists N such that $|x_n - x| < \eta$ whenever $n > N$. Now let η be of the form $\eta = C\epsilon$. The key point is

that (since $C > 0$ is fixed) to be given an arbitrary $\eta > 0$ is the same as being given an arbitrary $\epsilon > 0$ as we can get one from the other through the relations $\eta = C\epsilon$ and $\epsilon = \frac{C}{\eta}$. Hence we can rewrite the definition as stated in the question.

30. Since (a_n) is null, given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that if $n > N$, then $|a_n| < \epsilon$. Since (b_n) is bounded, there exists $C > 0$ so that $|b_n| \leq C$ for all $n \in \mathbb{N}$. Then for $n > N, |a_n b_n| = |a_n|.|b_n| < C\epsilon$, and the result follows by Problem 29. For a counter–example, take $a_n = 1$ and $b_n = (-1)^n$, for all $n \in \mathbb{N}$.

31. (a) Assume that $l < 0$. As (x_n) converges to l, given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that if $n > N$, then $l - \epsilon < x_n < l + \epsilon$. Now take $\epsilon = -l/2$. Then for all $n > N, x_n < l/2 < 0$, and that gives the desired contradiction.

    (b) The sequence whose nth term is $a - x_n$ converges to $a - x$ by the algebra of limits. Since $a - x_n > 0$ for all n, then we can use (a) to see that $a - x \geq 0$. The assertion is false. For a counter example take $x_n = 1 - 1/n < 1$ for all $n \in \mathbb{N}$, but the limit is 1.

32. If $x_n \to l$ then $x_{n+1} \to l$ as $n \to \infty$. So by algebra of limits, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$.

    (i) $\frac{1}{n!}$, (ii) $\frac{1}{2^n}$,

    (iii) For (x_n), consider the sequence that begins $1, 2, 1/2, 1, 1/3, 2/3, 1/4, 1/2, \ldots$ Two general terms in succession are $\frac{1}{n}, \frac{2}{n}$ and two general terms in succession of $\frac{x_{n+1}}{x_n}$ are $2, \frac{n}{2(n+1)}$. Then the odd terms of the sequence form a subsequence that has constant value 2, while the even terms form a subsequence that converges to $1/2$. So the sequence diverges.

33. (a) $r^{\frac{1}{n}} = 1 + c_n$ and so $r = (1 + c_n)^n \geq 1 + nc_n$. Hence $0 < c_n \leq \frac{r-1}{n}$, so by the sandwich rule, $c_n \to 0$ as $n \to \infty$ and the result follows.

    (b) $s > 1$ and $\lim_{n\to\infty} r^{\frac{1}{n}} = \frac{1}{\lim_{n\to\infty} s^{\frac{1}{n}}} = 1$ by algebra of limits and (a).

    (c) $(\sqrt{n})^{\frac{1}{n}} = 1 + c_n$ and so $\sqrt{n} = (1 + c_n)^n \geq 1 + nc_n$ as in (a). It follows that $0 < c_n \leq \frac{1}{\sqrt{n}} - \frac{1}{n}$. So by the sandwich rule, $c_n \to 0$ as $n \to \infty$. Then by the algebra of limits, $n^{\frac{1}{n}} = (\sqrt{n})^{\frac{1}{n}}.(\sqrt{n})^{\frac{1}{n}} \to 1$ as $n \to \infty$.

    Note. I learned the trick of dealing with $\sqrt{n}$ here from Courant and Robbins, "What is Mathematics?" Why doesn't the more obvious ploy: $n^{\frac{1}{n}} = 1 + c_n$ deliver the goods?