MAS221 Analysis (Semester 1)– Solutions to Problems 34 to 40

34. (a) Suppose that $x_n \geq 2$ for all $1 \leq n \leq m$ but $x_{m+1} < 2$. But $x_{m+1} = \frac{1}{5}(x_n^2 + 6) \geq \frac{1}{5}(4+6) \geq 2$ and we have a contradiction. The proof that $x_n \leq 3$ for all n is similar.

    Alternatively, prove by induction that $2 \leq x_n \leq 3$ for all $n \in \mathbb{N}$. Its true for $n = 1$. Suppose true for some n. Then $4 \leq x_n^2 \leq 9$ and so $2 \leq \frac{1}{5}(x_n^2 + 6) \leq 3$, i.e. $2 \leq x_{n+1} \leq 3$, and so the required inequality is true by induction for all $n \in \mathbb{N}$.

    (b) This is simple algebra.

    (c) By (b) we have $\frac{1}{5}(x_n - 2)(x_n - 3) \leq 0$ and so $x_{n+1} \leq x_n$, i.e. the sequence is monotonic decreasing. Let $\alpha = \lim_{n\to\infty} x_n$. Then from (a) and algebra of limits we get $0 = \alpha - \alpha = \frac{1}{5}(\alpha - 2)(\alpha - 3)$. So $\alpha = 2$ or 3. But the sequence is monotonic decreasing and $x_1 = 2.5$. It follows that $\alpha = 2$.

35. By Problem 15(b), given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $a_N < \inf_{n\in\mathbb{N}}(a_n) + \epsilon$. But since the sequence is monotonic decreasing, we deduce that $a_n \leq a_N < \inf_{n\in\mathbb{N}}(a_n) + \epsilon$ for all $n > N$, i.e.

    $$|a_n - \inf_{n\in\mathbb{N}}(a_n)| = a_n - \inf_{n\in\mathbb{N}}(a_n) < \epsilon.$$

    Alternative solution. First check that (a_n) is monotonic decreasing and bounded below if and only if $(-a_n)$ is monotonic increasing and bounded above. By Theorem 2.3.1 (1), $(-a_n)$ converges to $\sup_{n\in\mathbb{N}}(-a_n)$. The result follows from the fact that $\sup_{n\in\mathbb{N}}(-a_n) = -\inf_{n\in\mathbb{N}}(a_n)$ (see Theorem 1.4.3).

36. (a) To prove that $b_n \leq b_{n+1}$ look at

    $$\begin{aligned} b_{n+1}^2 - b_n^2 &= a_n b_n - a_{n-1}b_{n-1} \\ &= \frac{1}{2}(a_{n-1} + b_{n-1})\sqrt{a_{n-1}b_{n-1}} - a_{n-1}b_{n-1} \\ &= \sqrt{a_{n-1}b_{n-1}}\left(\frac{1}{2}(a_{n-1} + b_{n-1}) - \sqrt{a_{n-1}b_{n-1}}\right) \geq 0, \end{aligned}$$

    by the theorem of the means.

    (b) $a_{n+1} - b_{n+1} = \frac{1}{2}(a_n + b_n) - \sqrt{a_n b_n}$. Now since $b_n \leq a_n$, $b_n^2 \leq a_n b_n$ and so $b_n \leq \sqrt{a_n b_n}$. It follows that $a_{n+1} - b_{n+1} \leq \frac{1}{2}(a_n + b_n) - b_n = \frac{1}{2}(a_n - b_n)$.

(c) Deduce that (a_n) is monotonic decreasing and bounded below and that (b_n) is monotonic increasing and bounded above. To show that they have the same limit, first establish that $a_{n+1} - b_{n+1} \leq \frac{1}{2^n}(a-b)$.

37. Since (a_n) is a convergent sequence, it is bounded, and so is bounded above. Now by Theorem 2.3.1 it converges to its supremum. But we are given that it converges to l, and so by uniqueness of limits, $l = \sup_{n \in \mathbb{N}}(a_n)$. If (a_n) is monotonic decreasing and convergent to l, a similar argument shows that $l = \inf_{n \in \mathbb{N}}(a_n)$.

38. (a) $\frac{a_{n+1}}{a_n} = \frac{(n+1)^p x^{n+1}}{n^p x^p} = \left(1 + \frac{1}{n}\right)^p x \to x$ as $n \to \infty$.

   (b) The sequence whose nth term is $\left(1 + \frac{1}{n}\right)^p$ is monotonic decreasing to 1 so we can find $N \in \mathbb{N}$ such that for all $n > N$,

$$\left(1 + \frac{1}{n}\right)^p - 1 = \left|\left(1 + \frac{1}{n}\right)^p - 1\right| < \frac{1}{x} - 1$$

   and then $\frac{a_{n+1}}{a_n} < 1$ for all such n.

   (c) Define $b_n = a_{N+n}$. Then the sequence (b_n) is monotonic decreasing by (b). We have $b_n > 0$ for all n since $a_n > 0$ for all n so (b_n) is bounded below by 0. Hence it converges to l (say) by Theorem 2.3.1. But we then must have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+N} = l$, and $l \geq 0$ by Problem 29.

   (d) Suppose $l > 0$. Then by Problem 30, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. But by (a), $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ and we have a contradiction. So we must have $x = 0$.

   If $-1 < x < 0$, replace x by $|x|$. Then $\lim_{n\to\infty} n^p |x|^n \to 0$ by what we've already shown. But that is enough since given any $\epsilon > 0$, we can find $n \in \mathbb{N}$ such that if $n > N$ then $|n^p x^n - 0| = n^p |x|^n < \epsilon$.

39. (a) By the result of Problem 37, $a_{n_k} \leq l$ for all k. Now for any $n_k \leq n \leq n_{k+1}$ we have $a_n \leq a_{n_{k+1}} \leq l$ and the result follows.

   (b) Given any $\epsilon > 0$ we can find $K \in \mathbb{N}$ such that if $k > K$ then $l - \epsilon < a_{n_k} < l + \epsilon$. Then for $n > n_k, l - \epsilon < a_{n_k} < a_n \leq l < l + \epsilon$ and the result follows.

   (c) (a) changes to $a_n \geq l$ and (b) still holds.

40. (a) To see that (a_n) is decreasing consider $a_{n+1} = \sup(x_m; m \geq n+1)$. The sup is over the same list of numbers as appears in a_n with one exception; x_n is missing. But x_n may be larger than x_m for $m > n$. If it is then $a_{n+1} < a_n$. If it isn't then $a_{n+1} = a_n$. In either case

$a_{n+1} \leq a_n$. The sequence (a_n) is bounded below because (x_n) is, indeed $x_n \geq \inf_{n\in\mathbb{N}}(x_n)$ for all n and so $a_n = \sup(x_m; m \geq n) \geq \inf_{n\in\mathbb{N}}(x_n)$ for all n.

(b) This is a similar argument to (a).

(i) $\limsup 1, \liminf -1$, (ii) $\limsup 0, \liminf 0$, (iii) $\limsup 1, \liminf -1$.