MAS221 Analysis (Semester 1)– Solutions to Problems 41 to 46

41. Suppose that (a_n) is Cauchy. Then given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that if $m, n > N$ then $|a_m - a_n| < \epsilon$. Choose $\epsilon = 1$ (say). Then for any $n > N$, by Theorem 1.3.1,

$$|a_n| \leq |a_{N+1}| + |a_n - a_{N+1}| \leq 1 + |a_{N+1}|,$$

so the required bound is

$$K = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |a_{N+1}|\}.$$

42. Suppose that (a_n) converges to l, then given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $n > N$ implies $|a_n - l| < \epsilon/2$. Now take both $m, n > N$. Then by the triangle inequality,

$$|a_m - a_n| \leq |a_m - a| + |a - a_n| < \epsilon/2 + \epsilon/2 = \epsilon.$$

43. The sequence $(1/n)$ converges to 0, and so is Cauchy by Problem 42. But its limit does not lie in $(0, 1]$.

44. (a) $\mathbb{R} \setminus \{0, -1\}$, (b) $\mathbb{R} \setminus \{1, -2, -3\}$ (factorise the denominator to see this), (c) $\mathbb{R} \setminus \{-2, -3\}$, (d) $\mathbb{R} \setminus \{1\}$, (e) $\mathbb{R} \setminus \{0\}$.

45. For $x \neq 1$, $f_2(x) = \frac{(x+4)}{(x+2)(x+3)}$ from which $\lim_{x\to 1} f_2(x) = 5/12$.

    By considering sequences of the form $-2 + 1/n$ and $-3 + 1/n$ (respectively), we see that $\lim_{x\to -2} f_2(x)$ and $\lim_{x\to -3} f_2(x)$ don't exist.

46. The first part follows by using the definition of the limit of a function in terms of limits of sequences, and then applying the result of Problem 25 c).

    To be precise let (x_n) be any sequence in $\mathbb{R} \setminus \{a\}$ that converges to a. Then since we are given that $\lim_{x\to a} f(x) = l$, we must have that $\lim_{n\to\infty} f(x_n) = l$. But then by Problem 25 c), we have $\lim_{n\to\infty} \sqrt{f(x_n)} = \sqrt{l}$. So by definition of the limit of a function, $\lim_{x\to a} \sqrt{f(x)} = \sqrt{l}$.

    Then, using algebra of limits and the result just proved,

$$\lim_{x\to 1} \sqrt{\frac{x+1}{x^2}} = \sqrt{2}.$$