MAS221 Analysis (Semester 1)– Solutions to Problems 47 to 54

47. Suppose that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} f(x) = l'$, with $l \neq l'$. Then given any sequence (x_n) in $D_f \setminus \{a\}$ that converges to a, we have $\lim_{n\to\infty} f(x_n) = l$ and also $\lim_{n\to\infty} f(x_n) = l'$. But then $l = l'$ by Theorem 2.1.1 and we have our desired contradiction.

48. When $x > 0$, $\dfrac{|x|}{x} = \dfrac{x}{|x|} = \dfrac{x}{x} = 1 = \text{sgn}(x)$,

    when $x < 0$, $\dfrac{|x|}{x} = \dfrac{x}{|x|} = -\dfrac{x}{x} = -1 = \text{sgn}(x)$.

$$\lim_{x\uparrow 0} \text{sgn}(x) = -1, \lim_{x\downarrow 0} \text{sgn}(x) = 1.$$

49. (a) The only point at which left and right limits disagree is $x = 1$, with $\lim_{x\uparrow 1} f(x) = 0, \lim_{x\downarrow 1} f(x) = 1$.

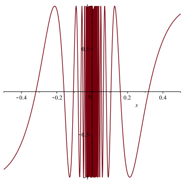
    (b) In this case, left and right limits disagree at every $n \in \mathbb{Z}$, with $\lim_{x\uparrow n}[x] = n - 1, \lim_{x\downarrow n}[x] = n$.

    (c) Here, left and right limits disagree at $x = 0, 1$ and 2. We have $\lim_{x\uparrow 0} h(x) = 3, \lim_{x\downarrow 0} h(x) = -2; \lim_{x\uparrow 1} h(x) = -2, \lim_{x\downarrow 1} h(x) = 10,;$

    $\lim_{x\uparrow 2} h(x) = 10, \lim_{x\downarrow 2} h(x) = 3$.

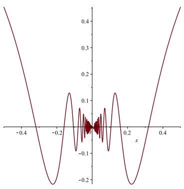
50. $D_f = \mathbb{R} \setminus \{0\}$. It has no limit at $x = 0$. To see this first take $\theta = \pi/2$, and consider the sequence (x_n) whose nth term is $\dfrac{1}{\pi/2 + 2\pi n}$. Then $\lim_{n\to\infty} x_n = 0$, but $\lim_{n\to\infty} f(x_n) = 1$. Next take $\theta = 3\pi/2$, and consider the sequence (y_n) whose nth term is $\dfrac{1}{3\pi/2 + 2\pi n}$. Then $\lim_{n\to\infty} y_n = 0$, but $\lim_{n\to\infty} f(y_n) = -1$. Here is how Maple draws the graph close to zero:

Figure 1: Graph of $f(x) = \sin(1/x)$



51. $D_f = \mathbb{R} \setminus \{0\}$. We have $\lim_{x \to 0} x \sin(1/x) = 0$. To see this let (x_n) be an arbitrary sequence in D_f that converges to 0. Then since $-1 \leq \sin(1/x_n) \leq 1$ for all $n \in \mathbb{N}$, the sequence whose nth term is $\sin(1/x_n)$ is bounded. Then the result follows by Problem 30. Alternatively use the sandwich rule, together with Problem 23: since $-1 \leq \sin(1/x_n) \leq 1$ then $-|x_n| \leq x_n \sin(1/x_n) \leq |x_n|$. You should contrast the picture below with that for the previous question.

Figure 2: Graph of $f(x) = x \sin(1/x)$



52. We'll just deal with the left limit here, as the right limit argument is very similar. Assume the (ϵ, δ)–criterion for left limits holds, and consider an arbitrary sequence (x_n) with $x_n < a$ for all $n \in \mathbb{N}$ that

converges to a. Then given any $\eta > 0$, there exists $N \in \mathbb{N}$ so that if $n > N$, then $0 < a - x_n < \eta$. Now choose η to be δ from the criterion, and argue as in the proof of Theorem 3.3.2.

The converse works in the same way as in that proof, except that this time the sequence (x_n) is constructed by choosing $x_n \in D_f$ such that $0 < a - x_n < 1/n$, and $|f(x_n) - l| \geq \epsilon$ for each $n \in \mathbb{N}$.

53. We'll just do $\lim_{x\to\infty} f(x)$ here, as $\lim_{x\to-\infty} f(x)$ is so similar.

(a) We say that $\lim_{x\to\infty} f(x) = l$ if whenever (x_n) is a sequence that diverges to infinity, with $x_n \in D_f$ for all $n \in \mathbb{N}$, then $\lim_{x\to\infty} f(x_n) = l$.

(b) The analogue of the $(\epsilon - \delta)$ criterion is $(\epsilon - K)$: given any $\epsilon > 0$, there exists $K > 0$ such that if $x > K$ then $|f(x) - l| < \epsilon$.

For the proof, suppose the $(\epsilon - K)$ criterion holds and $\lim_{n\to\infty} x_n = \infty$. Then given any $L > 0$, there exists $N \in \mathbb{N}$ so that if $n > N$, we have $x_n > L$. Now take L to be K from the criterion and we get that for all $n > N, |f(x_n) - l| < \epsilon$. Hence $\lim_{x\to\infty} f(x_n) = l$, as required.

For the converse, we again imitate the proof of Theorem 3.2.2. This time choose successively $K = 1, 2, \ldots$ and construct $x_n \in D_f$ such that $x_n > n$, and $|f(x_n) - l| \geq \epsilon$ for each $n \in \mathbb{N}$.

(c) Given any $\epsilon > 0$, choose $K = 1/\epsilon$. Then $x > K \Rightarrow 1/x < \epsilon$, and the result follows.

54. (a) We have $\lim_{x\to\infty} f(x) = \infty$ if for any sequence (x_n) (with $x_n \in D_f$ for all $n \in \mathbb{N}$) which diverges to infinity, we also have $(f(x_n))$ diverges to infinity. The analogue of $(\epsilon - \delta)$ is: given any $K > 0$, there exists $L > 0$ such that $x > L \Rightarrow f(x) > K$. The other cases are similar.

(b) Given any $K > 0$, there exists $L_1 > 0$ such that $x > L_1 \Rightarrow f(x) > K$, and given any $\epsilon > 0$ there exists $L_2 > 0$ so that $x > L_2 \Rightarrow l - \epsilon < g(x) < l + \epsilon$. Choose any $\epsilon < l$. Then for $x > \max\{L_1, L_2\}$ we have $f(x)g(x) > K(l - \epsilon)$, and the result follows. In the other case, we have $\lim_{x\to-\infty} h(x) = -\infty$; for in that case, given $K > 0$, there exists $L_1 > 0$ such that $x < -L_1 \Rightarrow f(x) < -K$, and for $x < \min\{-L_1, -L_2\}$ we have $f(x)g(x) < -K(l + \epsilon)$.

(c) Write $m = 2n$ and let

$$p(x) \quad = \quad a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$$

$$= x^{2n}\left(a_{2n} + \frac{a_{2n-1}}{x} + \cdots + \frac{a_1}{x^{2n-1}} + \frac{a_0}{x^{2n}}\right).$$

In part (b), take $f(x) = x^{2n}$ and $g(x) = a_{2n} + \frac{a_{2n-1}}{x} + \cdots + \frac{a_1}{x^{2n-1}} + \frac{a_0}{x^{2n}}$. Then $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = a_{2n}$, and the result follows from that of (b). If n is odd, $\lim_{x\to\infty} p(x) = \infty$, but $\lim_{x\to-\infty} p(x) = -\infty$.