MAS221 Analysis (Semester 1)– Solutions to Problems 55 to 65

55. For each of (a) to (e), the function is continuous at each point of its domain.

56. (a) i. By Theorem 1.3.1, if (x_n) is any sequence in D_f that converges to a,

$$\begin{aligned} 0 \leq ||f|(a) - |f|(x_n)| &= ||f(a) - |f(x_n||\\ &\leq |f(a) - f(x_n)| \to 0 \text{ as } n \to \infty, \end{aligned}$$

as f is continuous. So by the sandwich rule, $\lim_{n\to\infty} |f|(x_n) = |f|(a)$, and so f is continuous.

ii. By Problem 46 and (i), $\lim_{x\to a} \sqrt{|f|(x)} = \sqrt{|f|(a)}$.

(b) It is continuous on $(0, \infty)$ by (ii). The problem with $x = 0$ is a subtle one. The point 0 violates the domain condition mentioned in the footnote at the beginning of section 3.2. From another point of view, we cannot take left limits, as square roots of negative numbers are not real numbers. But you can check that $x \to \sqrt{x}$ is right continuous at $x = 0$.

57. Let (x_n) be a sequence in D_g that converges to a, then since g is continuous we have $\lim_{n\to\infty} g(x_n) = g(a)$. But f is continuous at $g(a)$ and so $\lim_{n\to\infty} f(g(x_n)) = f(g(a))$, and the result follows.

58. $(f \circ g)(x) = \frac{1}{1+x^2}$ for all $x \in \mathbb{R}$. It is continuous on $\mathbb{R}$ by Theorem 4.1.2 (4).

$(g \circ f)(x) = 1 + 1/x^2$ for all $x \in D_f = \mathbb{R} \setminus \{0\}$. It is continuous on $\mathbb{R} \setminus \{0\}$ by Theorem 4.1.2 (4).

59. (a) Continuous on $\mathbb{R} \setminus \{1\}$. Jump discontinuity at 1 with $J_f(1) = 1$.

(b) Continuous on $\mathbb{R} \setminus \mathbb{Z}$. Jump discontinuity at n with $J_g(n) = 1$ for all $n \in \mathbb{Z}$.

(c) Continuous at $\mathbb{R} \setminus \{0, 1, 2\}$. Each of $0, 1, 2$ is a jump discontinuity and we have $J_h(0) = -5, J_h(1) = 12, J_h(2) = -7$.

60. (a) For $x \neq 0, f(x) = x + 2$. Since $\lim_{x\to 0} f(x) = 2$, the required continuous extension is $\tilde{f}$ where

$$\tilde{f}(x) = \begin{cases} \dfrac{(1+x)^2 - 1}{x} & \text{if } x \neq 0 \\[2ex] 2 & \text{if } x = 0. \end{cases}$$

(b) The domain is $\mathbb{R} \setminus \{-2, 2\}$. For $x \neq 2, -2$, $f(x) = \dfrac{x^2 + 2x + 4}{x + 2}$, which is a rational function, and hence continuous on its domain. $\lim_{x \to -2} f(x)$ does not exist. To see this consider the sequence (x_n), whose nth term is $-2 + 1/n$, and check that

$$f(x_n) = n(-4 - 2/n + 1/n^2) \to -\infty \text{ when } n \to \infty$$

Thus f has no continuous extension to the point $x = -2$. But on the other hand, $\lim_{x \to 2} f(x) = 3$, and so f has a continuous extension $\tilde{f}$ to $\mathbb{R} \setminus \{-2\}$ given by

$$\tilde{f}(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2. \end{cases}$$

61. Assume $g(a) > 0$ and that the given statement is false. Then, in particular, for all $n \in \mathbb{N}$ there exists $x_n \in (a - 1/n, a + 1/n)$ so that $g(x_n) \leq 0$. By (for example) the sandwich rule, we have $\lim_{n \to \infty} x_n = a$. Then by continuity of g at a we have $g(a) = \lim_{n \to \infty} g(x_n) \leq 0$ (using Problem 31(b)), and we have our desired contradiction. The case $g(a) < 0$ is proved similarly, or just apply the result just proved to the function $-g$.

62. (a)

$$\max\{a, b\} = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } a < b. \end{cases}$$

On the other hand,

$$\frac{1}{2}(a + b) + \frac{1}{2}|a - b| = \begin{cases} \frac{1}{2}(a + b) + \frac{1}{2}(a - b) & = a \quad \text{if } a \geq b \\ \frac{1}{2}(a + b) + \frac{1}{2}(b - a) & = b \quad \text{if } a < b, \end{cases}$$

and the result follows.

Then for all $x \in D_f \cap D_g$,

$$\max\{f, g\}(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|,$$

and continuity follows by Theorem 4.1.2 (1), Problem 56(a)(i) and then Theorem 4.1.2 (1) again.

(b) Check that for all $a, b \in \mathbb{R}$,

$$\min\{a, b\} = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|,$$

and then argue as in (a), or derive (b) from (a) by using $\min\{f, g\} = -\max\{-f, -g\}$.

63. (a) $f(0) = f(0+0) = f(0) + f(0) = 2f(0)$, hence $f(0) = 0$.

(b) By (a), $0 = f(0) = f(x + -x) = f(x) + f(-x)$.

(c) If $a \neq 0$ then any sequence (x_n) which converges to a can be written as $x_n = a + y_n$ where (y_n) converges to zero. Then by (a)

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(a+y_n) = f(a) + \lim_{n\to\infty} f(y_n) = f(a) + f(0) = f(a).$$

(d) Assume $n \in \mathbb{N}$ and use induction. It is true for $n = 1$. Assume it holds for some $n \in \mathbb{N}$, then

$$f(n+1) = f(n) + f(1) = nk + k = (n+1)k.$$

If $n < 0$, apply (b) to the result just proved.

(e) By (d), if $p > 0, pk = f(p) = f(q.p/q) = qf(p/q)$, and the result follows. If $p < 0$, use (b).

(f) Suppose that x is irrational. Then we can find a sequence (p_n/q_n) of rational numbers that converges to x (see e.g. Example 4.4 in the notes). Then using (e), we have

$$f(x) = \lim_{n\to\infty} f(p_n/q_n) = k \lim_{n\to\infty} p_n/q_n = kx.$$

64. Suppose that $x \in \mathbb{Q}$. Then as in Example 4.4 we can find a sequence (x_n) of irrationals that converges to x and then

$$\lim_{n\to\infty} g(x_n) = 0 \neq g(x),$$

and so g cannot be continuous at x.

65 The function $\mathbf{1}_{(a,b)}$ is left continuous at a (but not right continuous), and right continuous at b (but not left continuous). To prove the left continuity, let (x_n) be any sequence in $\mathbb{R}$ which converges to a with $x_n < a$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \mathbf{1}_{(a,b)}(x_n) = 0 = \mathbf{1}_{(a,b)}(a)$. On the other hand to see that it is not right continuous, let (y_n) be any sequence in $\mathbb{R}$ which converges to a with $a < y_n < b$ for all $n \in \mathbb{N}$. Tnen $\lim_{n\to\infty} \mathbf{1}_{(a,b)}(y_n) = 1 \neq \mathbf{1}_{(a,b)}(a)$. The other assertion is proved similarly.

[Contrast this with $\mathbf{1}_{[a,b]}$, which was discussed in the notes.]