The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Metric Spaces **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Show that a sequence (x_n) in a metric space can have at most one limit. ***(4 marks)***

(ii) Let $n \geqslant 1$. Define the taxicab metric d_1 and the maximum or supremum metric d_∞ on $\mathbb{R}^n$. ***(2 marks)***

For each of the points $p = (\frac{7}{8}, \frac{1}{8}, \frac{9}{8})$ and $q = (\frac{7}{8}, 0, \frac{11}{8})$ of $\mathbb{R}^3$, determine whether it is in the open ball $B_1((1, 0, 1), \frac{3}{8})$ for the metric d_1 and whether it is in the open ball $B_\infty((1, 0, 1), \frac{3}{8})$ for the metric d_∞. ***(4 marks)***

Which, if any, of your answers would change if the open balls were replaced by the closed balls $B_1[(1, 0, 1), \frac{3}{8}]$ and $B_\infty[(1, 0, 1), \frac{3}{8}]$? ***(2 marks)***

(iii) Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots b_n)) \in \mathbb{R}^n$. Show that

$$d_\infty(a, b) \leqslant d_1(a, b) \leqslant nd_\infty(a, b)$$

and deduce that, for $r > 0$,

$$B_\infty(a, \tfrac{r}{n}) \subseteq B_1(a, r) \subseteq B_\infty(a, r).$$

(4 marks)

Let $f : X \to Y$ be a function between metric spaces. Explain, in terms of open balls, what it means for f to be *continuous*. ***(3 marks)***

Let X be a metric space and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function. Show that $f : (\mathbb{R}^n, d_1) \to (\mathbb{R}^n, d_1)$ is continuous if and only if $f : (\mathbb{R}^n, d_\infty) \to (\mathbb{R}^n, d_\infty)$ is continuous. ***(6 marks)***

2 The metrics d_1 and d_∞ on the space $C([0,1])$ of continuous functions from $[0,1]$ to $\mathbb{R}$ are given by the rules

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx, \quad d_\infty(f,g) = \sup_{0 \leqslant x \leqslant 1} |f(x) - g(x)|.$$

(i) Let (f_n) be any sequence of functions in $C([0,1])$ and let $f \in C([0,1])$. Show that if (f_n) converges to f in $(C([0,1]), d_\infty)$ then (f_n) converges to f in $(C([0,1]), d_1)$.
(5 marks)

(ii) For $n \geqslant 1$, let $f_n \in C([0,1])$ be given by the rules

$$f_n(x) = \begin{cases} x^2 + (n - \frac{1}{n})x \text{ if } 0 \leqslant x \leqslant \frac{1}{n} \\ 1 \text{ if } \frac{1}{n} \leqslant x \leqslant 1. \end{cases}$$

Let $f \in C([0,1])$ be given by the rule $f(x) = 1$ for all $x \in [0,1]$.

(a) Find $d_1(f_n, f)$ and $d_\infty(f_n, f)$ for $n \geqslant 1$. ***(7 marks)***

Deduce that $(f_n) \to f$ in $(C([0,1]), d_1)$ but that (f_n) does not converge to f in $(C([0,1]), d_\infty)$. ***(2 marks)***

(b) Show that the sets

$$S_1 = \{g \in C([0,1]) : g(0) = 0\} \text{ and } S_2 = \{g \in C([0,1]) : g(0) \neq 1\}$$

are not closed in $(C([0,1]), d_1)$. ***(3 marks)***

Is the set S_1 open in $(C([0,1]), d_1)$? Justify your answer. ***(4 marks)***

(c) Let $\theta : C([0,1]) \to C([0,1])$ be the identity function, $\theta(f) = f$ for all $f \in C([0,1])$. Using (i) and (a), explain why θ is continuous as a function from $(C([0,1]), d_\infty)$ to $(C([0,1]), d_1)$ but not as a function from $(C([0,1]), d_1)$ to $(C([0,1]), d_\infty)$.
(4 marks)

3 (i) Explain what it means for a subset A of a metric space to be *complete.* ***(2 marks)***

Show that in a complete metric space X every closed subset of X is complete. ***(3 marks)***

For each of the following subsets S_i of $\mathbb{R}$, determine whether S_i is complete under the usual metric on $\mathbb{R}$. Justify your answers. ***(7 marks)***

(a) $S_1 = (0, 1)$;

(b) $S_2 = \mathbb{R}\backslash\mathbb{Q}$ (the set of irrational numbers);

(c) $S_3 = \mathbb{R}\backslash S_1$.

(ii) Explain what it means for a subset A of a metric space to be *compact.* ***(2 marks)***

Show that in a compact metric space X every closed subset of X is compact. ***(3 marks)***

For each of the following subsets A_i of $\mathbb{R}^4$, determine whether A_i is compact under the Euclidean metric d_2. Justify your answers. ***(8 marks)***

(a) A_1 is the closed ball $B[(2, 0, 1, 5), 1]$;

(b) A_2 is the complement $\mathbb{R}^4\backslash B((2, 0, 1, 5), 1)$ of the open ball $B((2, 0, 1, 5), 1)$;

(c) A_3 is the intersection of all closed balls of the form $B[(x, x, x, x), 5]$, $0 \leqslant x \leqslant 1$.

4 (i) Let $f : X \to X$ be a function from a metric space to itself. Explain what it means for f to be a *contraction* on X. State, without proof, the Contraction Mapping Principle. ***(4 marks)***

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. State, without proof, a differential criterion for f to be a contraction on $\mathbb{R}$. ***(2 marks)***

Let a, b be positive real numbers. Show that the function $g(x) = ax\cos(bx)$ is not a contraction and that if $ab < 1$ then the function $f(x) = a\sin(bx) + 1$ is a contraction on $\mathbb{R}$. ***(6 marks)***

Show that there exists a unique real number x such that $3\sin(2x) = 7x - 7$. ***(3 marks)***

Explain briefly how you would calculate successively better approximate values for the number x. (Do not attempt the calculation!) ***(2 marks)***

(ii) Let X be a complete metric space and let $g : X \to X$ and $h : X \to X$ be contractions on X. Show that the composite function $g \circ h$ is a contraction on X. ***(4 marks)***

Explain why each of $g \circ h$ and $h \circ g$ has a unique fixed point in X and let y, z be the unique fixed points of $g \circ h$ and $h \circ g$ respectively. Show that $h(y) = z$ and $g(z) = y$. ***(4 marks)***

End of Question Paper