The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017–18**

MAS331 Metric Spaces **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

1 Let (X, d) be a metric space.

(i) Give precise definitions of the following.

(a) An open ball $B(a, r)$ in (X, d). ***(1 mark)***

(b) An open set in (X, d). ***(2 marks)***

(ii) Determine whether either of the points $(0, 0, 0, 0)$ or $(1/2, 1/4, 1/2, 1/4)$ are in the set $B((1, 0, 1, 0), 1) \cap B((0, 1, 0, 1), \sqrt{2})$ in $\mathbb{R}^4$, equipped with its usual Euclidean metric. ***(7 marks)***

(iii) Define the *taxi–cab metric* on $\mathbb{R}^m$ by

$$d_1(x, y) = \sum_{i=1}^{m} |x_i - y_i|,$$

where $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$.

(a) Show that $d_2(x, y) \leq d_1(x, y)$ for all $x, y \in \mathbb{R}^m$, where d_2 denotes the usual (Euclidean) metric on $\mathbb{R}^m$. ***(4 marks)***

(b) Show that $B_1(a, r) \subseteq B_2(a, r)$ for all $a \in \mathbb{R}^m, r > 0$, where B_1 denotes an open ball in $(\mathbb{R}^m, d_1)$, and B_2 denotes an open ball in $(\mathbb{R}^m, d_2)$. ***(3 marks)***

(iv) Prove that every open ball in a metric space is an open set. ***(6 marks)***

(v) Is it true that every open set in a metric space is an open ball? If so, give a proof. If not, give a counter–example. ***(2 marks)***

2 (i) Explain carefully what it means for a sequence (x_n) to converge to x in a metric space (X, d). Write down an equivalent statement in terms of convergence of a sequence of real numbers. ***(3 marks)***

(ii) Prove that the limit of a convergent sequence in a metric space is unique. ***(6 marks)***

(iii) Consider the metrics d_1 and d_∞ on $C[0,1]$ where

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx, \quad d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Show that if a sequence of functions converges in $(C[0,1]), d_\infty)$, then it converges to the same limit in $(C[0,1]), d_1)$. ***(4 marks)***

(iv) Consider the following sequences of functions in $C[0,1]$. Do they converge in either of $(C[0,1]), d_\infty)$ or $(C[0,1]), d_1)$, or pointwise, or none of these? Give reasoning to support your conclusions.

(a) $f_n(x) = \dfrac{3(1-x)^n}{5}$. ***(6 marks)***

(b) $f_n(x) = \cos\left(\dfrac{n - x^2}{n^2}\right)$. ***(6 marks)***

[Hint: You might find it helpful to use the fact that the mapping $x \to \cos(x)$ is monotonic decreasing on $[0, \pi/2]$, or alternatively to make use of the inequality $1 - \cos(x) \leq x^2/2$ for all $x \in \mathbb{R}$.]

3 (i) Let (X_1, d_1) and (X_2, d_2) be given metric spaces. Explain, using sequences, what it means for a mapping $f : X_1 \to X_2$ to be continuous. ***(2 marks)***

(ii) Let $\mathbb{R}^2$ and $\mathbb{R}$ have their usual metrics and define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F((x, y)) = e^{3y^2-4y+5} - g(x)^9,$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous. Show that F is continuous. ***(4 marks)***

(iii) With the notation used in (i), suppose that $f : X_1 \to X_2$ is continuous.

(a) Show that $f^{-1}(A)$ is closed in (X_1, d_1) whenever A is closed in (X_2, d_2). ***(4 marks)***

(b) Show that $f^{-1}(B)$ is open in (X_1, d_1) whenever B is open in (X_2, d_2). ***(2 marks)***

(iv) Which of the following sets are open, closed, both or neither in $(\mathbb{R}^3, d_2)$? You must present reasoning to support your conclusions. Here d_2 is the usual (Euclidean) metric on $\mathbb{R}^3$.

(a) $A_1 = \{(x, y, z) \in \mathbb{R}^3; -1 \leq x + y^2 + z^3 \leq 1\}$. ***(3 marks)***

(b) $A_2 = \{(x, y, z) \in \mathbb{R}^3; -1 < x + y^2 + z^3 < 1\}$. ***(2 marks)***

(c) $A_3 = \{(x, y, z) \in \mathbb{R}^3; -1 < x + y^2 + z^3 \leq 1\}$. ***(5 marks)***

(v) Let (X, d_1) and (Y, d_2) be metric spaces for which $X \subseteq Y$ and there exists $K > 0$ so that $d_2(x, y) \leq K d_1(x, y)$ for all $x, y \in X$. Show that the inclusion map $\theta : X \to Y$ is continuous, where $\theta(x) = x$ for all $x \in X$. ***(3 marks)***

4 (i) Let (X, d) be a metric space. Explain what it means for a mapping $f : X \to X$

(a) to be a *contraction*, ***(2 marks)***

(b) to have a *fixed point*. ***(1 mark)***

(ii) Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Fix $x_0 \in X$ and define a sequence (x_n) by $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$

(a) Deduce that there exists $0 \le k < 1$ such that for all $n = 0, 1, 2, \ldots$,

$$d(x_{n+1}, x_n) \le k^n d(x_1, x_0).$$

(4 marks)

(b) Prove that for $m > n$,

$$d(x_n, x_m) \le \frac{k^n}{1-k} d(x_1, x_0),$$

and hence show that (x_n) is a Cauchy sequence. ***(6 marks)***

(c) Explain why the sequence (x_n) has a limit x, and then show that x is the *unique* fixed point of f. ***(7 marks)***

(iii) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = \left(\frac{5}{3} - \frac{2y}{3}, \frac{1}{3} + \frac{2x}{3} \right).$$

Show that f is a contraction on $\mathbb{R}^2$ (with its usual metric). ***(5 marks)***

5 (i) Explain carefully what it means for a metric space to be (a) complete, (b) compact. ***(4 marks)***

(ii) Which of the following sets are either compact, complete, both or neither. Give reasoning to support your answer, quoting any results you need from the course.

(a) $[1, \infty)$ in $\mathbb{R}$ with its usual metric. ***(2 marks)***

(b) The closed ball $B[(0, 1, 0), 2]$ in $\mathbb{R}^3$ with its usual metric. ***(3 marks)***

(c) The set $\{f_1, f_2, f_3\}$ in $(C[0, 1], d_\infty)$, where $f_1(x) = 1 - x, f_2(x) = \sin(x)$ and $f_3(x) = \cos(2x)$. Here d_∞ is defined as in question 2. ***(3 marks)***

(iii) If (x_n) is a Cauchy sequence in a metric space (X, d) which has a subsequence (x_{n_j}) converging to x, show that the sequence (x_n) also converges to x. Hence prove that every compact metric space is complete. ***(8 marks)***

(iv) If C_1 and C_2 are compact sets in a metric space (X, d), show that $C_1 \cup C_2$ is also compact. ***(5 marks)***

End of Question Paper