Solutions to MAS 331/6352, Metric Spaces, 2017-18

1. (i) (a) $B(a,r) = \{x \in X; d(x,a) < r\}$,

(b) O is an open set if given any $a \in O$, there exists $r > 0$ such that $B(a,r) \subseteq O$.

(ii) $B((1,0,1,0),1) = \{(x,y,z,w) \in \mathbb{R}^4; (x-1)^2 + y^2 + (z-1)^2 + w^2 < 1\}$.

$B((0,1,0,1),\sqrt{2}) = \{(x,y,z,w) \in \mathbb{R}^4; x^2 + (y-1)^2 + z^2 + (w-1)^2 < 2\}$.

$(0,0,0,0) \notin B((1,0,1,0),1)$ as

$$(0-1)^2 + 0^2 + (0-1)^2 + 0^2 = 2 > 1,$$

so $(0,0,0,0) \notin B((1,0,1,0),1) \cap B((0,1,0,1),\sqrt{2})$.

(*Similar argument shows* $(0,0,0,0) \notin B((0,1,0,1),\sqrt{2})$.)

$(1/2,1/4,1/2,1/4) \in B((1,0,1,0),1) \cap B((0,1,0,1),\sqrt{2})$ as

$$(1/2-1)^2 + (1/4)^2 + (1/2-1)^2 + 1/4^2 = 5/8 < 1,$$

$$\text{and } (1/2)^2 + (1/4-1)^2 + (1/2)^2 + (1/4-1)^2 = 13/8 < 2.$$

(iii) (a)

$$\begin{aligned} d_1(x,y)^2 &= \left(\sum_{i=1}^m |x_i - y_i|\right)^2 \\ &= \sum_{i=1}^m (x_i - y_i)^2 + 2\sum_{i \neq j} |x_i - y_i||x_j - y_j| \\ &\geq \sum_{i=1}^m (x_i - y_i)^2 = d_2(x,y)^2. \end{aligned}$$

Hence $d_1(x,y) \geq d_2(x,y)$.

(b) If $x \in B_1(a,r)$ then $d_1(x,a) < r$. But by (a), we then have $d_2(x,a) < r$. Hence $x \in B_2(a,r)$ and the result follows.

(iv) Let $B(a,r)$ be the open ball under consideration and choose $y \in B(a,r)$. Thus $d(a,y) < r$.

Let $\epsilon := r - d(a,y) > 0$ and let $c \in B(y,\epsilon)$. So $d(y,c) < \epsilon$.

Then $d(a,c) \leq d(a,y) + d(y,c) < d(a,y) + \epsilon = r$.

Thus $c \in B(a,r)$.

Hence $B(y,\epsilon) \subseteq B(a,r)$, which is therefore open.

(v) No, e.g. consider the set $(0,1) \cup (2,3)$ in $\mathbb{R}$ with its usual metric.

2. (i) (x_n) converges to x if given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that if $n > N$ then $d(x_n, x) < \epsilon$.
   Equivalently, $\lim_{n\to\infty} d(x_n, x) = 0$.

   (ii) Suppose that (x_n) converges to both x and y with $y \neq x$.
   Given any $\epsilon > 0$, there exists $M \in \mathbb{N}$ so that if $n > M$, then $d(x_n, x) < \epsilon/2$, and there exists $N \in \mathbb{N}$ so that if $n > N$ then $d(x_n, y) < \epsilon/2$. Then for $n > \max\{M, N\}$,

$$\begin{aligned} d(x,y) &\leq d(x, x_n) + d(x_n, y) \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

   Since we can obtain this inequality for any $\epsilon > 0$, we deduce that $d(x,y) = 0$ and so $x = y$, which is the required contradiction.

   (iii) Suppose that (f_n) is a sequence of functions in $C[0,1])$ converging to f in the d_∞ metric. Since

$$\begin{aligned} 0 \leq d_1(f_n, f) &= \int_0^1 |f_n(x) - f(x)| dx \\ &\leq \sup_{x\in[0,1]} |f_n(x) - f(x)| \\ &= d_\infty(f_n, f) \to 0 \text{ as } n \to \infty, \end{aligned}$$

   we conclude (from e.g. the sandwich rule) that (f_n) converges to f in the d_1 metric.

   (iv) (a)

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= \frac{3}{5} \lim_{n\to\infty} (1-x)^n \\ &= \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 3/5 & \text{if } x = 0 \end{cases} \end{aligned}$$

   Since (f_n) does not converge pointwise to a continuous function, it does not converge in $(C[0,1], d_\infty)$. But

$$\begin{aligned} 0 \leq d_1(f_n, 0) &= \int_0^1 f_n(x) dx \\ &= \frac{3}{5} \int_0^1 (1-x)^n dx = \frac{3}{5(n+1)} \to 0 \text{ as } n \to \infty, \end{aligned}$$

   and so (f_n) converges in $(C[0,1], d_1)$.

(b) *Method 1* The mapping $x \to 1 - \cos\left(\frac{n-x^2}{n^2}\right)$ is continuous and monotonic decreasing on $[0,1]$ and so it attains its supremum at $x = 0$. Thus

$$d_\infty(f_n, 1) = \sup_{x\in[0,1]} \left|1 - \cos\left(\frac{n-x^2}{n^2}\right)\right| = 1 - \cos(1/n) \to 0 \text{ as } n \to \infty.$$

Hence (f_n) converges to the constant function 1, uniformly, and so pointwise and in d_1.

Method 2. Using the hint, for all $x \in [0,1]$,

$$1 - \cos\left(\frac{n-x^2}{n^2}\right) \leq \frac{1}{2}\left(\frac{n-x^2}{n^2}\right)^2 \leq \frac{1}{2n^2}$$

Then $d_\infty(f_n, 1) = \sup_{x\in[0,1]} \left|1 - \cos\left(\frac{n-x^2}{n^2}\right)\right| \leq \frac{1}{2n^2} \to 0$ as $n \to \infty$. Hence (f_n) converges to the constant function 1, uniformly, and so pointwise and in d_1.

3. (i) Given any sequence (x_n) that converges to a limit x in (X_1, d_1), then the sequence $(f(x_n))$ converges to $f(x)$ in (X_2, d_2).

(ii) Let $((x_n, y_n))$ converge to (x, y) in $\mathbb{R}^2$. Then by a theorem in the course, the sequences (x_n) and (y_n) converge to x and y in $\mathbb{R}$ (respectively). Now by continuity of the exponential function and algebra of limits, $e^{3y_n^2-4y_n+5} \to e^{3y^2-4y+5}$, and by continuity, $g(x_n)^9 \to g(x)^9$, as $n \to \infty$. Then

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} [e^{3y_n^2-4y_n+5} - g(x_n)^9] = e^{3y^2-4y+5} - g(x)^9 = F(x, y),$$

and so F is continuous.

(iii) (a) Let (x_n) be a sequence in $f^{-1}(A)$ that converges to some $x \in X_1$. Then by continuity $(f(x_n))$ converges to $f(x)$ in X_2. But $f(x_n) \in A$ for all $n \in \mathbb{N}$, and so $f(x) \in A$ as A is closed. But then $x \in f^{-1}(A)$ and so A is closed.

(b) If B is open, then its complement B^c is closed and since $f^{-1}(B^c) = f^{-1}(B)^c$, it follows that $f^{-1}(B) = (f^{-1}(B)^c)^c$ is open.

(iv) Use the fact that $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous where $f((x, y, z)) = x + y^2 + z^3$.

(a) $A_1 = f^{-1}([-1, 1])$ is closed by (iii)(a).

(b) $A_2 = f^{-1}((-1, 1))$ is open by (iii)(b).

(c) A_3 is not closed for consider the sequence (a_n) where $a_n = f((-1+1/n, 0, 0)) \in A_3$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = f((-1,0,0)) \notin A_3$. A_3 is not open for if it were, $A_3^c = \{(x,y,z) \in \mathbb{R}^3; x + y^2 + z^2 \leq -1\} \cup \{(x,y,z) \in \mathbb{R}^3; x + y^2 + z^2 > 1\}$ would be closed. Now consider the sequence (b_n) where $b_n = f((0,0,1+1/n)) \in A_3^c$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} b_n = f((0,0,1)) \notin A_3^c$.

(v) Let (x_n) be a sequence that converges to x in (X_1, d_1). Then

$$\begin{aligned} 0 \leq d_2(\theta(x_n), \theta(x)) &= d_2(x_n, x) \\ &\leq K d_1(x_n, x) \quad \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence θ is continuous, as required.

4. (i) (a) f is a contraction if there exists $0 \leq k < 1$ such that $d(f(x), f(y)) \leq kd(x,y)$ for all $x, y \in X$.

(b) f has a fixed point if there exists $x \in X$ such that $f(x) = x$.

(ii) (a) Since each $d(x_i, x_{i+1}) = d(f(x_{i-1}), f(x_i)) \leqslant kd(x_{i-1}, x_i)$, we have

$$d(x_{n+1}, x_n) \leqslant kd(x_n, x_{n-1}) \leqslant k^2 d(x_{n-1}, x_{n-2}) \leqslant \cdots \leqslant k^n d(x_1, x_0).$$

(b) By the triangle inequality

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leqslant [k^n + k^{n+1} + \cdots + k^{m-1}] d(x_0, x_1) \text{ by (a)} \\ &\leqslant [k^n + k^{n+1} + \cdots] d(x_0, x_1) \\ &= \frac{k^n}{1-k} d(x_0, x_1) \text{ (sum of a GP).} \end{aligned}$$

Since $k < 1$, it follows that $\frac{k^n}{1-k} \to 0$ as $n \to \infty$. It follows that (x_n) is a Cauchy sequence.

(c) The sequence has a limit since it is Cauchy and X is complete. To see that x is a fixed point, since contractions are continuous:

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x,$$

so that $f(x) = x$ as required.
For uniqueness, suppose that both x and x' are fixed points. Then

$$d(x, x') = d(f(x), f(x')) \leqslant kd(x, x').$$

As $k < 1$, the only way this can happen is if $d(x, x') = 0$, and so $x = x'$.

(iii)

$$\begin{aligned} d(f(x,y),f(x',y'))^2 &= \left(\frac{5}{3}-\frac{2y}{3}-\frac{5}{3}+\frac{2y'}{3}\right)^2+\left(\frac{1}{3}+\frac{2x}{3}-\frac{1}{3}-\frac{2x'}{3}\right)^2 \\ &= \frac{4}{9}(y-y')^2+\frac{4}{9}(x-x')^2 \\ &= \frac{4}{9}d((x,y),(x',y'))^2 \end{aligned}$$

and so f is a contraction with $k = 2/3$.

5. (i) (a) A metric space (X, d) is complete if every Cauchy sequence in it converges to a limit in X.

   (b) A metric space (X, d) is compact if every sequence in it has a convergent subsequence.

   (ii) (a) $[1, \infty)$ is complete as it is a closed subset of $\mathbb{R}$ with its usual metric. It is not compact as the sequence (n) of natural numbers has no convergent subsequence. Alternatively it is closed but not bounded.

   (b) It is complete as it is a closed subset of $\mathbb{R}^3$ with its usual metric. It is compact by the Heine–Borel theorem, as it is closed and bounded.

   (c) *The set specified here is a red herring. Any finite set in any metric space is compact by this argument. Either* let (g_n) be an arbitrary sequence taken from the set. Then at least one of the f_i's must appear countably many times, and selecting these gives the required convergent subsequence, *or* choose an arbitrary open cover (U_α) of the set. Then for each $i \in \{1, 2, 3\}$, there exists U_i with $f_i \in U_i$ and $\{U_1, U_2, U_3\}$ is the required finite subcover (and in fact, one or more of these three sets might be the same).
   The set is complete because it is compact (by a theorem in the course).

   (iii) Since (x_n) is Cauchy, given any $\epsilon > 0$ there exists $M \in \mathbb{N}$ so that if $m, n > M$ then $d(x_m, x_n) < \epsilon/2$. As the subsequence converges, for the given value of ϵ, there exists $K \in \mathbb{N}$ so that if $k > K$, then $d(x_{n_k}, x) < \epsilon/2$. Now take $N > \max\{M, K\}$ and put $m = n_N \geq N > M$. Then for $n > N$,

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_m) + d(x_m, x) \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Now if X is compact and (x_n) is a Cauchy sequence in X, it has a subsequence converging to x (say), and then (x_n) converges to X, so X is complete.

(iv) *Either* let (x_n) be a sequence in $C_1 \cup C_2$. Then a countable number of the x_n's must live in one of the C_i's - say C_1.Call this new sequence (y_n). Then it has a convergent subsequence in $C_1 \subseteq C_1 \cup C_2$. It follows that $C_1 \cup C_2$ is compact.

Or Let (U_α) be an open cover for $C_1 \cup C_2$. Then it is an open cover for both $C_1 \subseteq C_1 \cup C_2$ and $C_2 \subseteq C_1 \cup C_2$. But C_1 is compact, so there is a finite subcover $\{V_1, \ldots, V_N\}$ of C_1, and C_2 is compact, so there is a finite subcover $\{W_1, \ldots, W_M\}$ of C_2. But then $\{V_1, \ldots, V_N, W_1, \ldots, W_M\}$ is a finite subcover of $C_1 \cup C_2$, which is therefore compact.