MAS331 2014-15 Exam Solutions

MAS331 2014-15 Solution 1

(i) Suppose that $x_n \to a$ and $x_n \to b$. By axiom M3,

$$0 \leqslant d(a,b) \leqslant d(a,x_n) + d(x_n,b)$$

for all n.

Now since $x_n \to a$ and $x_n \to b$, $d(a,x_n) \to 0$ and $d(x_n,b) \to 0$.

By the algebra of limits, $d(a,x_n) + d(x_n,b) \to 0$ and, by the Sandwich Rule, the constant sequence $d(a,b) \to 0$. Hence $d(a,b) = 0$, and, by Axiom M1, $a = b$.

(ii) The *taxicab metric* d_1 on $\mathbb{R}^n$ is given by

$$d_1((a_1,\dots,a_n),(b_1,\dots,b_n)) = |a_1 - b_1| + \cdots + |a_n - b_n|.$$

The *maximum metric* or *supremum metric* d_∞ on $\mathbb{R}^n$ is given by the rule

$$d_\infty((a_1,\dots,a_n),(b_1,\dots,b_n)) = \max\{|a_1 - b_1|,\dots,|a_n - b_n|\}.$$

$$d_1((1,0,1),p) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

so $p \notin B_1((1,0,1),\frac{3}{8})$ and

$$d_\infty((1,0,1),p) = \max(\frac{1}{8},\frac{1}{8},\frac{1}{8}) = \frac{1}{8}$$

so $p \in B_\infty((1,0,1),\frac{3}{8})$.

$$d_1((1,0,1),q) = \frac{1}{8} + 0 + \frac{3}{8} = \frac{1}{2}$$

so $q \notin B_1((1,0,1),\frac{3}{8})$ and

$$d_\infty((1,0,1),q) = \max(\frac{1}{8},0,\frac{3}{8}) = \frac{3}{8}$$

so $q \notin B_\infty((1,0,1),\frac{3}{8})$.

The answers for p and d_1 and for q and d_∞ would change, as $d_1((1,0,1),p) = \frac{3}{8} = d_\infty((1,0,1),q)$ but the answers for the others would not.

(iii)

$$\begin{aligned} d_\infty(a,b) &= \max_{1\le i\le n} |a_i - b_i| \\ &\le \sum_{1\le i\le n} |a_i - b_i| \end{aligned}$$

because the maximum is one of the terms in the sum and the others are $\geqslant 0$. Thus $d_\infty(a,b) \le d_1(a,b)$.

$$\begin{aligned} d_1(a,b) &= \sum_{1\le i\le n} |a_i - b_i| \\ &\le n \max_{1\le i\le n} |a_i - b_i| \end{aligned}$$

because each of the n terms in the sum is $\le$ their maximum. Thus $d_1(a,b) \le n d_\infty(a,b)$.

Let $x \in B_\infty(a, \frac{r}{n})$. Thus $d_\infty(a,x) < \frac{r}{n}$. From above, $d_1(a,x) < r$ so $x \in B_1(a,r)$. Thus $B_\infty(a,\frac{r}{n}) \subseteq B_1(a,r)$.

Now let $x \in B_1(a,r)$. Thus $d_1(a,x) < r$. From above, $d_\infty(a,x) < r$ so $x \in B_\infty(a,r)$. Thus $B_1(a,r) \subseteq B_\infty(a,r)$.

A function $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is *continuous at* $x \in X$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(x,\delta)) \subseteq B(f(x),\epsilon)$.

f is *continuous* if f is continuous at every $x \in X$.

Suppose that $f : (\mathbb{R}^n, d_1) \to (\mathbb{R}^n, d_1)$ is continuous. Let $x \in \mathbb{R}^n$ and let $\epsilon > 0$. There exists $\delta > 0$ such that $f(B_1(x,\delta)) \subseteq B_1(f(x),\epsilon)$. Then

$$f(B_\infty(x, \frac{\delta}{n})) \subseteq f(B_1(x,\delta)) \subseteq B_1(f(x),\epsilon) \subseteq B_\infty(f(x),\epsilon).$$

Thus $f : (\mathbb{R}^n, d_\infty) \to (\mathbb{R}^n, d_\infty)$ is continuous.

Conversely, suppose that $f : (\mathbb{R}^n, d_\infty) \to (\mathbb{R}^n, d_\infty)$ is continuous. Let $x \in \mathbb{R}^n$ and let $\epsilon > 0$. There exists $\delta > 0$ such that $f(B_\infty(x,\delta)) \subseteq B_\infty(f(x),\frac{\epsilon}{n})$. Then

$$f(B_1(x,\delta)) \subseteq f(B_\infty(x,\delta)) \subseteq B_\infty(f(x),\frac{\epsilon}{n}) \subseteq B_1(f(x),\epsilon).$$

Thus $f : (\mathbb{R}^n, d_\infty) \to (\mathbb{R}^n, d_\infty)$ is continuous.

MAS331 2014-15 Solution 2

(i) Suppose that f_n converges to f in $(C[a,b], d_\infty)$. For each n, let $k_n = d_\infty(f_n, f)$. Then $k_n \to 0$ and $|f_n(x) - f| \le k_n$ whenever $0 \leqslant x \leqslant 1$.

Then

$$d_1(f_n, f) = \int_0^1 |f_n(x) - f(x)| dx < \int_0^1 k_n dx = k_n.$$

By the algebra of limits and the Sandwich Rule, $d_1(f_n, f) \to 0$ so $f_n \to f$ in $(C[a,b], d_1)$.

(ii) (a) Let $h(x) = f_n(x) - f(x) = x^2 + (n - \frac{1}{n})x - 1$. Then $h'(x) = 2x + (n - \frac{1}{n}) > 0$ for $x \ge 0$. Thus h is increasing from $h(0) = -1$ to $h(\frac{1}{n}) = 0$ on $[0, \frac{1}{n}]$. Thus $|f_n(x) - f(x)| = -x^2 - (n - \frac{1}{n})x + 1$ is decreasing from 1 to 0 on $[0, \frac{1}{n}]$.

Therefore $d_\infty(f_n, f) = 1$.

Also

$$\begin{aligned} & d_1(f_n,f) \\ = & \int_0^{\frac{1}{n}} (-x^2-(n-\tfrac{1}{n})x+1)dx+0 \\ = & \left[-\frac{x^3}{3}-\frac{1}{2}(n-\tfrac{1}{n})x^2+x\right]_0^{\frac{1}{n}} \\ = & -\frac{1}{3n^3}-\frac{1}{2n^2}(n-\tfrac{1}{n})+\frac{1}{n} \\ = & \frac{1}{6n^3}+\frac{1}{2n}. \end{aligned}$$

As $d_1(f_n,f)=\frac{1}{6n^3}+\frac{1}{2n}\to 0$, $(f_n)\to f$ in $(C([0,1]),d_1)$.

As $d_\infty(f_n,f)=1\not\to 0$, (f_n) does not converge to f in $(C([0,1]),d_\infty)$.

(b) Each $f_n\in S_1$, because $f_n(x)=0$, but $f\notin S_1$, because $f(0)=1\neq 0$. Therefore S_1 is not closed.

Each $f_n\in S_2$, because $f_n(x)=0\neq 1$, but $f\notin S_2$, because $f(0)=1$. Therefore S_2 is not closed.

S_1 is open if and only if the complement $S_3=\{g\in C([0,1]):g(0)\neq 0\}$ is closed. Let

$$g_n(x)=1-f_n(x)=\begin{cases} -x^2-(n-\frac{1}{n})x+1 \text{ if } 0\leq x\leq \frac{1}{n} \\ 0 \text{ if } \frac{1}{n}\leq x\leq 1. \end{cases}$$

and let $g(x)=0$ for all x. Then $(g_n)\to g$ under d_1, because $d_\infty(g_n,g)=\int_0^{\frac{1}{n}}(-x^2-(n-\frac{1}{n})x+1)dx\to 0$. But each $g_n\in S_3$, because $g_n(0)=1\neq 0$, and $g\notin S_3$ so S_3 is not closed and S_1 is not open.

(c) A function $f\colon X\to Y$ between two metric spaces (X,d_X) and (Y,d_Y) is continuous at $x\in X$ if and only if whenever we have a sequence $x_1,x_2,\ldots$ of elements of X converging to x, then the sequence $f(x_1),f(x_2),\ldots$ in Y converges to the limit $f(x)\in Y$.

By (i), if $(f_n)\to f$ in (f_n) converges to f in $C([0,1]),d_\infty)$ then (f_n) converges to f in $(C([0,1]),d_1)$, that is $(\theta(f_n))$ converges to $\theta(f)$ in $(C([0,1]),d_1)$. Thus θ is continuous as a function from $(C([0,1]),d_\infty)$ to $(C([0,1]),d_1)$.

The sequence f_n and the function f in (ii) are such that (f_n) converges to f in $C([0,1]),d_1)$ but does not converge to f in $(C([0,1]),d_\infty)$, that is $((\theta(f_n)))$ does not converge to $\theta(f)$ in $(C([0,1]),d_\infty)$. Thus θ is not continuous as a function from $(C([0,1]),d_1)$ to $(C([0,1]),d_\infty)$.

MAS331 2014-15 Solution 3

(i) A subset $A\subseteq X$ is *complete* if every Cauchy sequence in A converges to a limit in A.

Let $a_1,a_2,\ldots$ be a Cauchy sequence in A. Then it is also a Cauchy sequence in X,

and therefore converges to some $x\in X$.

Since A is a closed subset of X, $x\in A$, so A is complete.

(ii)(a) $S_2=(0,1)$ is not complete.

The sequence $(\frac{1}{2n})$ in $(0,1)$ converges in $\mathbb{R}$ and is therefore Cauchy.

The limit $0 \notin (0,1)$ so $(0,1)$ is not complete.

(b) $S_3 = \mathbb{R}\backslash\mathbb{Q}$ is not complete.

The sequence $(1+\frac{\pi}{n})$ in S_2 converges in $\mathbb{R}$ and is therefore Cauchy. The limit $1 \notin S_2$ so S_2 is not complete.

(c) $S_3 = \mathbb{R}\backslash S_1$ is complete.

S_3 is closed and $\mathbb{R}$ is complete so, by the result in (i), S_3 is complete.

(iii) A is *compact* if every sequence in A has a subsequence that converges to a point of A.

Let a_n be a sequence in A. Then (a_n) is a sequence in X and, as X is compact, there is a subsequence (a_{n_k}) converging to a limit $a \in X$.

But each $a_{n_k} \in A$ and A is closed so $a \in A$. Therefore A is compact.

(a,b,c) If $A \subseteq \mathbb{R}^n$ then A is compact if and only if A is closed and bounded.

(a) A_1 is compact.

A_1 is closed and, as $d(a,b) \leq d(a,(2,0,1,5)) + d((2,0,1,5),b) \leq 2$ for all $a,b \in A_1$, bounded. Therefore A_1 is compact.

(b) A_2 is not compact.

Given $D > 0$, $(2, D+1, 1, 5) \in A_2$ and $d(2, D+1, 1, 5), (2,0,1,5)) = D+1 > D$. So A_2 is not bounded and therefore not compact.

(c) A_3 is compact.

A_3 is closed because the intersection of any set of closed sets is closed. A_3 is bounded because it is contained in $B[(0,0,0,0),5]$, so $d(a,b) \leq d(a,(0,0,0,0)) + d((0,0,0,0),b) \leq 10$ for all $a,b \in A_3$. Therefore A_3 is compact.

MAS331 2014-15 Solution 4

(i) f is a *contraction*, with contraction factor k, if there exists a constant $0 \leqslant k < 1$ such that

$$d(f(x), f(y)) \leqslant kd(x,y)$$

for all $x, y \in X$.

Contraction Mapping Principle: Let $f : X \to X$ be a contraction of the complete metric space (X,d). Then f has a unique fixed point.

Let $f \colon \mathbb{R} \to \mathbb{R}$ be differentiable. Then f is a contraction of $\mathbb{R}$ if and only if there exists $k < 1$ with $|f'(x)| \leqslant k$ for all $x \in \mathbb{R}$.

$g'(x) = a\cos(bx) - abx\sin(bx)$. Let $N \in \mathbb{Z}$. Then $|g'(2N\pi + \frac{\pi}{2})| = ab(2N\pi + \frac{\pi}{2})$. We can choose N such that $2N\pi + \frac{\pi}{2} > \frac{1}{ab}$ so that $|g'(x)| > 1$. By the differential criterion, g is not a contraction.

$|f'(x)| = |ab\cos(bx)| \leq ab$ for all $x \in \mathbb{R}$. By the differential criterion with $k = ab < 1$, f is a contraction.

Take $a = \frac{3}{7}$ and $b = 2$. By the Contraction Mapping Principle and as $\mathbb{R}$ is complete,

there is a unique $x \in \mathbb{R}$ such that $x = f(x) = \frac{3}{7}\sin(2x) + 1$. But $x = f(x) = \frac{3}{7}\sin(2x) + 1 \Leftrightarrow 7x - 7 = 3\sin(2x)$ so there is a unique $x \in \mathbb{R}$ such that $7x - 7 = 3\sin(2x)$.

Choose any starting value $y \in \mathbb{R}$ and calculate successive terms of the sequence $y, f(y), f^2(y), \ldots,$ which converges to x.

Let $k, \ell < 1$ be such that

$$d(g(x), g(y)) \leqslant kd(x, y) \text{ and } d(h(x), h(y)) \leqslant \ell d(x, y)$$

for all $x, y \in X$. Let $x, y \in X$. As h is a contraction, $d(h(x), h(y)) \leqslant \ell d(x, y)$.

As g is a contraction,

$$d(g(h(x)), g(h(y))) \leqslant kd(h(x), h(y)) \leqslant k\ell d(x, y).$$

That is $d(g \circ h(x), g \circ h(y)) \leqslant k\ell d(x, y)$ for all $x, y \in \mathbb{R}$.

Also $k\ell < \ell < 1$ as $k, \ell < 1$.

So $g \circ h$ is a contraction with contraction factor $k\ell$.

By the Contraction Mapping Principle, each of $g \circ h$ and $h \circ g$ has a unique fixed point in X.

As y is a fixed point of $g \circ h$, $h(y) = h(g \circ h(y)) = h(g(h(y))) = h \circ g(h(y))$. Thus $h(y)$ is a fixed point of $h \circ g$ and, by uniqueness of z, $h(y) = z$.

As z is a fixed point of $h \circ g$, $g(z) = g(h \circ g(z)) = g(h(g(z))) = g \circ h(g(z))$. Thus $g(z)$ is a fixed point of $g \circ h$ and, by uniqueness of y, $g(z) = y$. (or appeal to symmetry)