The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015–16**

MAS331 Metric Spaces **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) (a) The *supremum metric* d_∞ on $C[a,b]$ is defined by

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| \,:\, x \in I\}.$$

We must verify the three axioms for $(C(I), d_\infty)$ to be a metric space.

We begin with M1. Let $f, g \in C(I)$, so that f and g are continuous functions from I to $\mathbb{R}$. Then $d_\infty(f,g) = \sup\{|f(x) - g(x)| \,:\, x \in I\}$ is non-negative because each $|f(x) - g(x)|$ is non-negative. Also, $d_\infty(f,g)$ is 0 if and only if each $|f(x) - g(x)|$ is 0, which is if and only if $f(x) = g(x)$ for all x, which is if and only if $f = g$. This proves M1.

M2 is immediate because $|f(x) - g(x)| = |g(x) - f(x)|$ for all x, so that

$$\begin{aligned} d_\infty(f,g) &= \sup\{|f(x) - g(x)| \,:\, x \in I\} \\ &= \sup\{|g(x) - f(x)| \,:\, x \in I\} \\ &= d_\infty(g,f). \end{aligned}$$

Now we will prove M3. Let $h\colon I \to \mathbb{R}$ be a third element of $C(I)$, and note that

$$|f(x)-h(x)| = |(f(x)-g(x))+(g(x)-h(x))| \leq |f(x)-g(x)|+|g(x)-h(x)|$$

for all $x \in I$. Now we have

$$\begin{aligned} d_\infty(f,h) &= \sup\{|f(x) - h(x)| \,:\, x \in I\} \\ &\leq \sup\{|f(x) - g(x)| + |g(x) - h(x)| \,:\, x \in I\} \\ &\leq \sup\{|f(x) - g(x)| \,:\, x \in I\} + \sup\{|g(x) - h(x)| \,:\, x \in I\} \\ &= d_\infty(f,g) + d_\infty(g,h), \end{aligned}$$

as required.

(b) Let $x \in [a,b]$. We must prove that $f_n(x) \to f(x)$. So let $\epsilon > 0$ and let N be such that $d_\infty(f_n, f) < \epsilon$ for all $n > N$. This means that $\sup\{|f_n(t) - f(t)| \,:\, t \in [a,b]\} < \epsilon$ for all $n > N$, so that in particular $d(f_n(x), f(x)) = |f_n(x) - f(x)| < \epsilon$ for all $n > N$, and so $f_n(x) \to f(x)$ as required.

(0 mark)

 Question 1 continued on next page

1 (continued)

(ii) (a) We say that F is a *closed* subset of X if: Whenever we have a sequence $x_1, x_2, \ldots$ in F which converges to a limit $a \in X$, then the limit a also lies in F.

(b) Suppose that $f_n \to f$ and that $f_n \in D_{a,b}$ for all n.
Then for any x we have $f_n(x) \to f(x)$ by (i)(b) and by definition of $D_{a,b}$ we have

$$a(x) \leq f_n(x) \leq b(x)$$

for all n.
Taking limits, it follows that

$$a(x) \leq f(x) \leq b(x).$$

This holds for all x, and so $f \in D_{a,b}$.

(iii) Let $a(x) = -x^2$ and $b(x) = x^2$. Then by definition we have $f_n \in D_{a,b}$ for all n, and so $f \in D_{a,b}$, i.e.

$$-x^2 \leq f(x) \leq x^2$$

for all x. It follows that

$$\int_0^1 -x^2 dx \leq \int_0^1 f(x) dx \leq \int_0^1 x^2 dx$$

which simplifies to

$$-1/3 \leq \int_0^1 f(x) dx \leq 1/3$$

as required.

2 (i) (a) Let $x_n \to a$ and $x_n \to b$. We will show that $a = b$. Let $\epsilon > 0$.
Since $x_n \to a$ there is N_1 such that $d(x_n, a) < \epsilon/2$ for $n \geqslant N_1$.
Since $x_n \to b$ there is N_2 such that $d(x_n, b) < \epsilon/2$ for $n \geqslant N_2$.
Then, we choose $n \geqslant \max(N_1, N_2)$, and observe

$$d(a,b) \leqslant d(a,x_n) + d(x_n,b) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since ϵ was any choice of positive number, the only possibility is that $d(a,b) = 0$.
Then $a = b$ from the axioms of a metric space.

(b) Let $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ be any subsequence of (x_n).
Let $\epsilon > 0$.
Since we know that $x_n \to a$, there is a number N such that if $n > N$, $d(x_n, a) < \epsilon$.
We choose K so that $n_K \geqslant N$.
Then as long as $k > K$, $n_k > n_K \geqslant N$, and we conclude that $d(x_{n_k}, a) < \epsilon$.

(ii) (a) We note that

$$\begin{aligned} x_{4k} &= \cos(2k\pi + \frac{\pi}{8k}) \\ &= \cos(\frac{\pi}{8k}) \\ &\to 1 \end{aligned}$$

since cos is continuous, since $\pi/8k \to 0$.

(b)

$$\begin{aligned} x_{4k+2} &= \cos((2k+1)\pi + \frac{\pi}{8k+4}) \\ &= -\cos(\frac{\pi}{8k+4}) \\ &\to -1 \end{aligned}$$

by the hint, since $\pi/(8k+4) \to 0$.

(c) If (x_n) converged to a limit x then both x_{4k} and x_{4k+2} would converge to that limit as well, by (i)(b). But then $x = 1$ by (ii)(a) and $x = -1$ by (ii)(b), a contradiction.

(iii) Let x_{n_k} be a subsequence such that $x_{n_k} \to x$.
Then there is a K such that $|x_{n_k} - x| < 1/2$ for all $k > K$, but then

$$1 \leq |x_{n_k} - x_{n_{k+1}}| \leq |x_{n_k} - x| + |x - x_{n_{k+1}}| < 1/2 + 1/2 = 1$$

and this is a contradiction.

3 (i) (a) Suppose that $(x_n, y_n) \to (x, y)$. We will show that $x_n \to x$. Let $\epsilon > 0$. Since $(x_n, y_n) \to (x, y)$ there is N such that $d_2((x_n, y_n), (x, y)) < \epsilon$ for $n \geqslant N$. But then

$$|x_n - x| \leqslant \sqrt{(x_n - x)^2 + (y_n - y)^2} = d_2((x_n, y_n), (x, y)) < \epsilon$$

for $n \geqslant N$, so that $x_n \to x$ as required. The proof that $y_n \to y$ is identical.

Now suppose that $x_n \to x$ and $y_n \to y$. We will show that $(x_n, y_n) \to (x, y)$. Let $\epsilon > 0$. Then since $x_n \to x$ there is N_1 such that $|x_n - x| < \epsilon/2$ for $n \geqslant N_1$. And since $y_n \to y$ there is N_2 such that $|y_n - y| < \epsilon/2$ for $n \geqslant N_2$. It follows that, for $n \geqslant \max(N_1, N_2)$ we have

$$\begin{aligned} d_2((x_n, y_n), (x, y)) &= \sqrt{(x_n - x)^2 + (y_n - y)^2} \\ &\leqslant \sqrt{(\epsilon/2)^2 + (\epsilon/2)^2} \\ &= \epsilon/\sqrt{2} \\ &< \epsilon \end{aligned}$$

so that $(x_n, y_n) \to (x, y)$ as required.

(b) By (i)(a) it suffices to show that $\frac{n-1}{n+1} \to 1$ and that $\frac{1}{n} \to 0$.

We show that $\frac{n-1}{n+1} \to 1$. Let $\epsilon > 0$ and set $N = 2/\epsilon$. Then for $n \geqslant N$ we have

$$\left|\frac{n-1}{n+1} - 1\right| = \frac{2}{n+1} \leqslant \frac{2}{N+1} < \frac{2}{N} = \epsilon.$$

This shows $\frac{n-1}{n+1} \to 1$.

Now we show that $\frac{1}{n} \to 0$. Let $\epsilon > 0$ and set $N = 1/\epsilon$. Then for $n > N$ we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \epsilon$$

so that $\frac{1}{n} \to 0$ as required.

 Question 3 continued on next page

3 (continued)

(ii) (a) A function $f\colon X \to Y$ is continuous if, for any sequence (x_n) in X that converges to a point x, the sequence $(f(x_n))$ in Y converges to the point $f(x)$.

(b) f is continuous if and only if, for each sequence $x_n \to x$ in $\mathbb{R}$, the sequence $f(x_n) \to f(x)$ in $\mathbb{R}^2$. But $f(x_n) = (f_1(x_n), f_2(x_n))$ and $f(x) = (f_1(x), f_2(x))$, so that by part (i)(a) we have that $f(x_n) \to f(x)$ if and only if $f_1(x_n) \to f_1(x)$ and $f_2(x_n) \to f_2(x)$. But f_1 and f_2 are continuous if and only if, for each $x_n \to x$, $f_1(x_n) \to f_1(x)$ and $f_2(x_n) \to f_2(x)$.

Combining all these facts, we have that f is continuous if and only if f_1 and f_2 are continuous.

4 (i) (a) We say that a sequence $x_1, x_2, \ldots$ in a metric space X is a *Cauchy sequence* if: For all $\epsilon > 0$, we can find N such that $d(x_m, x_n) < \epsilon$ whenever $m, n > N$.

(b) (X, d) is complete if every Cauchy sequence in X converges to some limit in X.

(c) $(C[0,1], d_\infty)$ is complete but $(C[0,1], d_1)$ is not.

(ii) (a) For $m \geqslant n$ we have

$$\begin{aligned} d_\infty(f_n, f_m) &= \sup\{|f_n(x) - f_m(x)| : x \in [0,1]\} \\ &= \sup\left\{\left|\frac{x^n}{2^n} + \cdots + \frac{x^m}{2^m}\right| : x \in [0,1]\right\} \\ &= \frac{1}{2^n} + \cdots + \frac{1}{2^m} \\ &= \frac{1}{2^{n-1}}\left(1 - \frac{1}{2^{m-n+1}}\right) \end{aligned}$$

(b) Let $\epsilon > 0$. Since $\frac{1}{2^{n-1}} \to 0$ there is N such that $\frac{1}{2^{n-1}} < \epsilon$ for $n \geqslant N$. Then for $m \geq n \geqslant N$,

$$d_\infty(f_m, f_n) = \frac{1}{2^{n-1}}\left(1 - \frac{1}{2^{m-n+1}}\right) < \frac{1}{2^{n-1}} \leqslant \epsilon$$

and so (f_n) is Cauchy.

(c) Since $(C[0,1], d_\infty)$ is complete, the sequence converges.

(iii) (a) .

(b) Let $m \geqslant n$. Then

$$\begin{aligned} d_1(g_n, g_m) &= \int_0^1 |g_n(x) - g_m(x)| dx \\ &= \int_0^1 g_n(x) - g_m(x) dx \\ &= \int_0^1 g_n(x) dx - \int_0^1 g_m(x) dx \end{aligned}$$

by the hint. But $\int_0^1 g_n(x)dx = \frac{1}{2} \times \frac{1}{2n} + \frac{1}{2}$, so that $d_1(g_n, g_m) = \frac{1}{4n} + \frac{1}{2} - \frac{1}{4m} - \frac{1}{2} = \frac{1}{4n} - \frac{1}{4m}$ as required.

(c) Let $\epsilon > 0$ and set $N = \frac{1}{4\epsilon}$. Then for $n, m \geqslant N$ we may assume that $m \geqslant n$, so that $d_1(g_n, g_m) = \frac{1}{4n} - \frac{1}{4m} < \frac{1}{4n} \leqslant \frac{1}{4N} = \epsilon$, so that (g_n) is Cauchy as required.

5 (i) (a) A contraction is a map $f\colon (X,d) \to (X,d)$ with the property that there is $0 \leqslant k < 1$ such that $d(f(x), f(y)) < kd(x,y)$ for all $x, y \in X$.

(b) The contraction mapping principle states that a contraction of a complete metric space has a unique fixed point.

Choose $x_0 \in X$, and define (x_n) inductively by iterating

$$x_{n+1} = f(x_n).$$

We will prove that (x_n) is a Cauchy sequence – as X is complete, the sequence will converge.

First we'll bound $d(x_n, x_{n+1})$. Note that each $d(x_i, x_{i+1}) = d(f(x_{i-1}), f(x_i)) \leqslant kd(x_{i-1}, x_i)$, so that

$$d(x_n, x_{n+1}) \leqslant kd(x_n, x_{n-1}) \leqslant k^2 d(x_{n-1}, x_{n-2}) \leqslant \cdots \leqslant k^n d(x_1, x_0).$$

Now we'll suppose that $m > n$ and find a bound for $d(x_n, x_m)$:

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leqslant [k^n + k^{n+1} + \cdots + k^{m-1}]d(x_0, x_1) \\ &\leqslant [k^n + k^{n+1} + \cdots]d(x_0, x_1) \\ &= \frac{k^n}{1-k} d(x_0, x_1). \end{aligned}$$

Since $k < 1$, it follows that $\dfrac{k^n}{1-k} \to 0$ as $n \to \infty$. It follows that (x_n) is a Cauchy sequence, and since X is complete, it follows that $x_n \to x$ for some limit $x \in X$.

Let's now see that x is a fixed point. Note that, since the sequence $x_0, x_1, \ldots$ converges to x, then the sequence $x_1, x_2, \ldots$ also converges to x. Therefore:

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x,$$

so that $f(x) = x$ as required.

Now we'll show that the fixed point is unique. Suppose that both x and x' are fixed points. Then

$$d(x, x') = d(f(x), f(x')) \leqslant kd(x, x').$$

As $k < 1$, the only way this can happen is if $d(x, x') = 0$, and so $x = x'$.

 Question 5 continued on next page

5 (continued)

(ii) (a) Note that

$$f(x) - f(y) = x + 1/x - y - 1/y = (x - y)\left(1 - \frac{1}{xy}\right)$$

so that

$$|f(x) - f(y)| = |x - y| \times \left(1 - \frac{1}{xy}\right) < |x - y|.$$

(b) A fixed point is an x such that $f(x) = x$, i.e. that $1/x = 0$, which can never happen.

(c) f reduces distances but is not a contraction, so there is no contradiction.

End of Question Paper