Solutions to MAS350 Exam 2015-16

1. (i) For (I) we need $A_n \in \Sigma$ for all $n \in \mathbb{N}$. But we also need $S \in \Sigma$ and $A^c \in \Sigma$ whenever $A \in \Sigma$.

   For (II) we need to replace $m : S \to [0, \infty)$ with $m : \Sigma \to [0, \infty]$ and $m(\emptyset) = 0$. We also need that the sequence (A_n) is such that $A_n \in \Sigma$ for all $n \in \mathbb{N}$, and that the sets in (A_n) are mutually disjoint. Suppose (A_n) is a sequence of sets in $\Sigma_1 \cap \Sigma_2$. Then $A_n \in \Sigma_1$ for all $n \in \mathbb{N}$ and so $\bigcup_{n=1}^{\infty} A_n \in \Sigma_1$. But also $A_n \in \Sigma_2$ for all $n \in \mathbb{N}$ and so $\bigcup_{n=1}^{\infty} A_n \in \Sigma_2$. Hence $\bigcup_{n=1}^{\infty} A_n \in \Sigma_1 \cap \Sigma_2$. If $A \in \Sigma_1 \cap \Sigma_2, A^c \in \Sigma_1$ and $A^c \in \Sigma_2$. Hence $A^c \in \Sigma_1 \cap \Sigma_2$.

   (b) $\Sigma_1 \cup \Sigma_2$ is not in general a σ-algebra for if $A \in \Sigma_1$ and $B \in \Sigma_2$ there is no good reason why $A \cup B \in \Sigma_1 \cup \Sigma_2$. For example let $S = \{1, 2, 3\}, \Sigma_1 = \{\emptyset, \{1\}, \{2, 3\}, S\}, \Sigma_2 = \{\emptyset, \{2\}, \{1, 3\}, S\}, A = \{1\}, B = \{2\}$. Then $A \cup B = \{1, 2\}$ is neither in Σ_1 nor Σ_2.

   (iii) (a)

$$\begin{aligned} A \triangle B &= (A \cup B) - (A \cap B) \\ &= (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) \cap (A^c \cup B^c) \\ &= [(A \cup B) \cap A^c] \cup [(A \cup B) \cap B^c] \\ &= (A \cap A^c) \cup (B \cap A^c) \cup (A \cap B^c) \cup (B \cap B^c) \\ &= (B \cap A^c) \cup (A \cap B^c) = (B - A) \cup (A - B). \end{aligned}$$

   (b) $B - A = (-1/3, 0] \cup [1, 2)$ and $A - B = [1/2, 3/4]$. The three intervals are mutually disjoint, and so $\lambda(A \triangle B) = 1/3 + 1 + 1/4 = 19/12$.

   (c) $\lambda(S) = 3$ and $P(A \triangle B) = \lambda(A \triangle B)/3 = 19/36$.

2. (i) To show that $f^{-1}((a, \infty)) \in \Sigma \Rightarrow f^{-1}([a, \infty)) \in \Sigma$ use $[a, \infty) = \bigcap_{n=1}^{\infty}(a - 1/n, \infty)$ and so

$$f^{-1}([a, \infty)) = \bigcap_{n=1}^{\infty} f^{-1}((a - 1/n, \infty))$$

   and the result follows since Σ is closed under countable intersections.

   To show that $f^{-1}([a, \infty)) \in \Sigma \Rightarrow f^{-1}((a, \infty)) \in \Sigma$, use

$$f^{-1}((a, \infty)) = \bigcup_{n=1}^{\infty} f^{-1}([a + 1/n, \infty))$$

and the fact that Σ is closed under countable unions.

(ii) *Either* use the fact that $\{a\} \in \mathcal{B}(\mathbb{R})$ together with the fact that f is measurable and so $f^{-1}(A) \in \Sigma$, for all $A \in \mathcal{B}(\mathbb{R})$.

or Use $\{a\} = \bigcap_{n=1}^{\infty} [a, a+1/n) \in \mathcal{B}(\mathbb{R})$ and argue as in (i).

(iii) For all $a \in \mathbb{R}, (f \circ g)^{-1}((a,\infty)) = g^{-1}(f^{-1}(a,\infty)$. Now f is Borel measurable and so $f^{-1}((a,\infty)) = A \in \mathcal{B}(\mathbb{R})$. Hence $g^{-1}(A) \in \Sigma$. So we conclude that $(g \circ f)^{-1}((a,\infty)) \in \Sigma$ and so $g \circ f$ is measurable.

(iv) Write $h = f \circ \tau_y$ where $\tau_y(x) = x + y$ for all $x \in \mathbb{R}$. The mapping τ_y is continuous and hence measurable and so h is measurable by (iii).

(v) If f is differentiable then it is continuous and so measurable. For each $x \in \mathbb{R}$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Now $x \to f(x+h)$ is measurable by (iv), and so $x \to \frac{f(x+h)-f(x)}{h}$ is measurable since sums and scalar multiples of measurable functions are. Finally f' is measurable since it is a pointwise limit of measurable functions.

(vi) f and $f^{(1)}$ are measurable by (v). Then for all $r = 2, \ldots, n$, if $f^{(r-1)}$ is measurable, for each $x \in \mathbb{R}$,

$$f^{(r)}(x) = \lim_{h \to 0} \frac{f^{(r-1)}(x+h) - f^{(r-1)}(x)}{h},$$

and so $f^{(r)}$ is also measurable by the argument of (v), and the result follows.

(vii) We use the fact that $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable functions, and that $\lim_{n\to\infty} f_n(x)$ exists for some $x \in S$ if and only if $\limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$, in which case $\lim_{n\to\infty} f_n(x)$ is their common value. So

$$A = \{x \in S; \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)\} = g^{-1}(\{0\},$$

where $g = \limsup_{n\to\infty} f_n - \liminf_{n\to\infty} f_n$ is measurable. The result follows by (ii).

3. (i) (a) If $f = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}$ where $n \in \mathbb{N}, c_i \geq 0$ and $A_i \in \Sigma$, for all $i = 1, \ldots, n$ being mutually disjoint with $\bigcup_{i=1}^{n} A_i = S$, then $\int_S f dm = \sum_{i=1}^{n} c_i m(A_i) \in [0,\infty]$.

(b) $\int_S f dm = \sup\left\{\int_S g dm; g \text{ simple } 0 \leq g \leq f\right\} \in [0, \infty]$.

(c) Write $f = f_+ - f_-$, where $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. Then $\int_S f dm = \int_S f_+ dm - \int_S f_- dm \in [-\infty, \infty]$, with the restriction that both integrals on the right hand side are not infinite.

f is integrable if $\int_S f dm \in (-\infty, \infty)$ (equivalently $\int_S |f| dm < \infty$).

(ii) (a) $f_+ = 4\mathbf{1}_{[-1,0)} + 11\mathbf{1}_{[0,1)}$, $f_- = 7\mathbf{1}_{[-2,1)} + 3\mathbf{1}_{[1,2)} + 2\mathbf{1}_{[2,5)}$.

(b) Since $|f| = f_+ + f_-$, we have

$$\begin{aligned} \int_{\mathbb{R}} |f| d\lambda &= \int_{\mathbb{R}} f_+ d\lambda + \int_{\mathbb{R}} f_- d\lambda \\ &= (4 + 11) + (7 + 3 + [2 \times 3)] = 31 \end{aligned}$$

(iii) g is measurable as it is a product of measurable functions. We have $\left|\frac{x}{1+x^2}\right| = \frac{|x|}{1+x^2} \leq \frac{1}{2}$, for since $(1 - |x|)^2 \geq 0$, we have $2|x| \leq 1 + x^2$. But then we have $|g(x)| \leq \frac{1}{2}|f(x)|$ for all $x \in \mathbb{R}$, **(1)** and so g is integrable as by monotonicity,

$$\int_{\mathbb{R}} |g| d\lambda \leq \frac{1}{2} \int_{\mathbb{R}} |f| d\lambda < \infty.$$

(iv) Since $g > 0$, the function $1/g$ is continuous, hence measurable and $\int_0^\infty 1/g(x)dx$ is well defined, and takes values in $[0, \infty]$. Using monotonicity, and the hint, for all $n \in \mathbb{N}$ we have

$$\int_1^\infty \frac{1}{g(x)} dx \geq \int_1^n \frac{1}{g(x)} dx = \int_1^n \sum_{k=2}^n \frac{1}{g(x)} \mathbf{1}_{[k-1,k)}(x) dx.$$

But on the interval $[k-1, k), 1/g(x) \geq 1/k$ and so

$$\begin{aligned} \int_1^\infty \frac{1}{g(x)} dx &\geq \int_1^n \sum_{k=2}^n \frac{1}{k} \mathbf{1}_{[k-1,k)}(x) dx \\ &= \sum_{k=2}^n \frac{1}{k} \int_1^n \mathbf{1}_{[k-1,k)}(x) dx \\ &= \sum_{k=2}^n \frac{1}{k} = \infty. \end{aligned}$$

4. (i) Let (f_n) be a sequence of measurable functions from S to $\mathbb{R}$ which converge pointwise to a (measurable) function f. Suppose there

is an integrable function $g : S \to \mathbb{R}$ so that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then f is integrable and

$$\int_S f dm = \lim_{n\to\infty} \int_S f_n dm.$$

(*You can also assume that f_n is integrable for all $n \in \mathbb{N}$, but that is not strictly necessary as it is established within the proof of the theorem.*)

(ii) Since $\frac{n}{e^{x^2}+n} \leq 1$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, we can use dominated convergence to prove the first result taking $g = |f|$ and $f_n = \frac{n}{e^{x^2}+n} f$.

For the second result, it follows from what has just been proved and linearity that $\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \left(1 - \frac{n}{e^{x^2}+n}\right) dx = 0$ and so

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{e^{x^2}}{e^{x^2}+n} f(x) dx = 0.$$

Alternatively use dominated convergence again as in the first part.

(iii) (a)

$$\frac{\partial}{\partial t} \int_S f(t,x) dm(x) = \lim_{c\to 0} \int_S \frac{f(t+c,x) - f(t,x)}{c} dm(x).$$

By the mean value theorem, for each $x \in S$ there exists $0 < \theta(x) < 1$ so that $\frac{f(t+c,x)-f(t,x)}{c} = \frac{\partial f}{\partial t}(t + \theta(x)c, x)$ and by assumption, $\left|\frac{\partial f}{\partial t}(t + \theta(x)c, x)\right| \leq h(x)$, where h is integrable. So using the dominated convergence theorem (where we implicitly replace c by an arbitrary sequence (c_n)), we obtain

$$\begin{aligned} \lim_{c\to 0} \int_S \frac{f(t+c,x) - f(t,x)}{c} dm(x) &= \int_S \lim_{c\to 0} \frac{f(t+c,x) - f(t,x)}{c} dm(x) \\ &= \int_S \frac{\partial f(t,x)}{\partial t} dm(x). \end{aligned}$$

(b) Implementing (a), define $g(t,x) = \sin(2tx^2)\frac{f(x)}{x^2}$. Then (I) $x \to g(t,x)$ is integrable since $\left|\sin(2tx^2)\frac{f(x)}{x^2}\right| \leq \left|\frac{f(x)}{x^2}\right|$. For (II) we have $\frac{\partial g(t,x)}{\partial t} = 2\cos(2tx^2)f(x)$, and (III) follows since $|\cos(2tx^2)f(x)| \leq |f(x)|$. By (a) we conclude that

$$\frac{\partial}{\partial t} \int_{[1,\infty)} \sin(2tx^2)\frac{f(x)}{x^2} dx = 2 \int_{[1,\infty)} \cos(2tx^2) f(x) dx.$$

(iv)

$$\int_S |f - f_n| d\lambda = \int_S |f \mathbf{1}_{(n,\infty)} d\lambda.$$

We have $|f\mathbf{1}_{(n,\infty)}| \leq |f|$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \mathbf{1}_{(n,\infty)}(x) = 0$ for all $n \in \mathbb{N}$, since given any $x \geq 0$ there exists $N \in \mathbb{N}$ such that $N > x$.[1] The result then follows by the dominated convergence theorem.

5. (i) (a) $\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k$ **(1)**, $\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k=n}^{\infty} A_k$. Both sets are in $\mathcal{F}$ as the σ–algebra is closed under finite and countable unions and intersections.

(b) As the events $\bigcup_{k=n}^{\infty} A_k$ form a decreasing sequence, by continuity of probability we have

$$\begin{aligned} P\left(\limsup_{n\to\infty} A_n\right) &= \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \\ &= \limsup_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \\ &\geq \limsup_{n\to\infty} P(A_n), \end{aligned}$$

where the last line is by monotonicity.

(c)

$$\begin{aligned} B - \liminf_{n\to\infty} A_n &= B \cap \left(\bigcup_{n\in\mathbb{N}} \bigcap_{k=n}^{\infty} A_k\right)^c \\ &= B \cap \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k^c \\ &= \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} (B \cap A_k^c) \\ &= \limsup_{n\to\infty} (B - A_n) \end{aligned}$$

For the last part, take $B = \Omega$.

(ii) Write $\mathbb{E}(\min\{X, a\}) = \int_\Omega \min\{X(\omega), a\} dP(\omega)$.
Now for each $\omega \in \Omega, \min\{X(\omega), a\} \leq a$ and $\min\{X(\omega), a\} \leq X(\omega)$, so by monotonicity, $\mathbb{E}(\min\{X, a\}) \leq \int_\Omega a dP(\omega) = 1$ and $\mathbb{E}(\min\{X, a\}) \leq \int_\Omega X(\omega) dP(\omega) = \mathbb{E}(X)$. Hence $\mathbb{E}(\min\{X, a\}) \leq \min\{E(X), a\})$.

[1] This is the Archimedean property of $\mathbb{R}$, but they don't need to say that.

(a) $\mathbb{E}(X) = 3/4$ so $\mathbb{E}(\min\{X, a\}) \le 3/4$.

(b) $\mathbb{E}(X) = \sum_{i=1}^{10} \mathbb{E}(Y_i) = 1 + 2 + \cdots + 10 = \frac{1}{2}.10.11 = 55$. Hence $\mathbb{E}(\min\{X, a\}) \le 54$.