MAS451/6352 Measure and Probability - Solutions to Additional Exercises on Product Measures and Fubini's Theorem I

1. (a)

$$\begin{aligned} (E \cap F)_x &= \{y \in S_2; (x,y) \in E \cap F\} \\ &= \{y \in S_2; (x,y) \in E\} \cap \{y \in S_2; (x,y) \in F\} \\ &= E_x \cap F_x. \end{aligned}$$

(b)

$$\begin{aligned} (E^c)_x &= \{y \in S_2; (x,y) \in E^c\} \\ &= \{y \in S_2; (x,y) \notin E\} \\ &= (E_x)^c. \end{aligned}$$

(c)

$$\begin{aligned} \left(\bigcup_{n=1}^{\infty} E_n\right)_x &= \left\{y \in S_2; (x,y) \in \bigcup_{n=1}^{\infty} E_n\right\} \\ &= \bigcup_{n=1}^{\infty} \{y \in S_2; (x,y) \in E_n\} \\ &= \bigcup_{n=1}^{\infty} (E_n)_x. \end{aligned}$$

2. We can write $S_1 = \bigcup_{n=1}^{\infty} A_n$ where $m_1(A_n) < \infty$ for all $n \in \mathbb{N}$ and $S_2 = \bigcup_{r=1}^{\infty} B_r$ where $m_2(B_r) < \infty$ for all $r \in \mathbb{N}$. We then have

$$S_1 \times S_2 = \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{\infty} A_n \times B_r,$$

and for all $r, n \in \mathbb{N}$,

$$(m_1 \times m_2)(A_n \times B_r) = m_1(A_n) m_2(B_r) < \infty.$$

(You can, of course, write $S_1 \times S_2$ as just a single union, by using the countability of $\mathbb{N} \times \mathbb{N}$.)