MAS451/6352 Measure and Probability - Solutions to Additional Exercises on Product Measures and Fubini's Theorem II

(3) Let $E = A \times B$. Then if $x \in S_1$,

$$E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

So $\phi_E(x) = m_2(B)\mathbf{1}_A(x)$, and hence

$$\begin{aligned}(m_1 \times m_2)(A \times B) &= \int_{S_1} \phi_E(x) dm_1(x) \\ &= m_2(B) \int_{S_1} \mathbf{1}_A(x) dm_1(x) \\ &= m_1(A) m_2(B).\end{aligned}$$

(4) Suppose that μ is a measure that takes the same value as $m_1 \times m_2$ on finite product sets. Define

$$\mathcal{E} = \{E \in \Sigma_1 \otimes \Sigma_2; \mu(E) = (m_1 \times m_2)(E)\}.$$

By definition of μ, the collection $\mathcal{P}$ of all finite product sets is in $\mathcal{E}$. Since

$$(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2),$$

it follows that $\mathcal{P}$ is a π-system. Using basic properties of measures, it is not hard to show that $\mathcal{E}$ is a λ-system (use the solution to Problem 1 to establish (L1)). By σ-finiteness, it follows that $\sigma(\mathcal{P}) = \Sigma_1 \otimes \Sigma_2$ and by Dynkin's $\pi - \lambda$ lemma, $\sigma(\mathcal{P}) \subseteq \mathcal{E}$. The result follows.

(5) (a) *Method 1.* First let $f = \mathbf{1}_A$ and let $g = \mathbf{1}_B$. Since $h = \mathbf{1}_A \mathbf{1}_B = \mathbf{1}_{A \times B}$, it is clear that h is measurable in this case. Next use linearity, to extend to the case where f and g are non-negative simple functions. Next let f and g be arbitrary non-negative measurable functions. Then by Theorem 2.4.1, there is a sequence of non-negative simple functions (s_n) converging pointwise to f, and a corresponding sequence (t_m) converging pointwise to g. Taking limits as m and n go to infinity, proves that f and g are measurable in this case. Finally let f and g be arbitrary measurable functions. Write $f = f_+ - f_-$ and $g = g_+ - g_-$. Then

$$fg = (f_+ g_+ + f_- g_-) - (f_- g_+ + f_+ g_-),$$

is measurable as it is a sum of products of measurable functions.

Method 2. For $B \in \Sigma_2$, define $\tilde{f}_B(x,y) = f(x)\mathbf{1}_B(y)$ for all $x \in S_1, y \in S_2$. The mapping $\tilde{f} : S_1 \times S_2 \to \mathbb{R}$ is measurable since for all $a \in \mathbb{R}$, $\tilde{f}^{-1}((a,\infty)) = f^{-1}((a,\infty)) \times B \in \Sigma_1 \times \Sigma_2$. In particular, $\tilde{f}_{S_2}$ is measurable; however $\tilde{f}_{S_2}(x,y) = f(x)$ for all $x \in S_1, y \in S_2$; so $h = \tilde{f}_{S_2}\tilde{g}_{S_1}$ is the product of measurable functions, hence is measurable.

(b) Follows easily from Fubini's theorem (2).

(6) Let m be counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Then $a(i,j) = a_{ij}$ defines a non-negative measurable function from $(\mathbb{N}^2, \mathcal{P}(\mathbb{N}^2))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where we note that $\mathcal{P}(\mathbb{N}^2) = \mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N})$. We have

$$\int_{\mathbb{N}^2} a \; d(m \times m) = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j},$$

$$\int_{\mathbb{N}} \left(\int_{\mathbb{N}} a(i,j) dm(i) \right) dm(j) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij},$$

$$\int_{\mathbb{N}} \left(\int_{\mathbb{N}} a(i,j) dm(j) \right) dm(i) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij},$$

and the result follows by Fubini's theorem 1.

(7) (a)

$$\begin{aligned} A_f^c &= \{(x,t) \in S \times \mathbb{R}; 0 \le f(x) < t\} \\ &= \bigcup_{q\in\mathbb{Q}} \{(x,t) \in S \times \mathbb{R}; 0 \le f(x) < q, t \ge q\} \\ &= \bigcup_{q\in\mathbb{Q}} f^{-1}([0,q)) \times (q,\infty), \end{aligned}$$

which is a countable union of measurable sets, and so is measurable. Hence $A_f = (A_f^c)^c$ is measurable.

(b) We use the definition (as a Lebesgue integral) of product measure. Fix $x \in S$. Then the x-slice $(A_f)_x$ is just the interval $[0, f(x)]$. Its Lebesgue measure is $f(x)$ and so

$$\begin{aligned} (m \times \lambda)(A_f) &= \int_S \lambda[(A_f)_x] dm(x) \\ &= \int_S f(x) dm(x). \end{aligned}$$

(8) First fix $T > 0$ and use the hint:

$$\int_0^T \frac{\sin(x)}{x} dx = \int_0^T \sin(x) \left(\int_0^\infty e^{-xy} dy \right) dx.$$

Now $f(x, y) = e^{-xy} \sin(x)$ is continuous, and so Riemann integrable (and hence Lebesgue integrable) on $[0, t] \times [0, N]$. By Fubini's theorem:

$$\begin{aligned} \int_0^T \sin(x) \left(\int_0^N e^{-xy} dy \right) dx &= \int_0^N \left(\int_0^T e^{-xy} \sin(x) dx \right) dy \\ &= -\int_0^N \left(\frac{y}{1+y^2} e^{-yT} \sin(T) \right. \\ &+ \left. \frac{1}{1+y^2} (e^{-yT} \cos(T) - 1) \right) dy, \end{aligned}$$

using integration by parts.

On the other hand, $\int_0^N e^{-xy} dy = \frac{1}{x}(1 - e^{-Ny})$ and so

$$\left| \int_0^N e^{-xy} dy \right| \leq \frac{2}{x}.$$

Since $x \to \frac{\sin(x)}{x}$ is continuous, and hence integrable, on $[0, T]$ we can use dominated convergence to assert that

$$\begin{aligned} \int_0^T \frac{\sin(x)}{x} dx &= \int_0^T \sin(x) \left(\int_0^\infty e^{-xy} dy \right) dx \\ &= -\int_0^\infty \left(\frac{y}{1+y^2} e^{-yT} \sin(T) + \frac{1}{1+y^2} (e^{-yT} \cos(T) - 1) \right) dy. \end{aligned}$$

Now use monotonicity in the first integral (since $y/1 + y^2 \leq 1$), and dominated convergence in the second (since $|e^{-yT} \cos(T) - 1| \leq 2$), to deduce that

$$\lim_{T \to \infty} \int_0^T \frac{\sin(x)}{x} dx = \int_0^\infty \frac{1}{1+y^2} dy = \frac{\pi}{2}.$$

(9) (a) Both integrals vanish by elementary calculus arguments.

(b) Let $S = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1, -1 \leq y \leq 1\}$ and $A = \{(x, y) \in \mathbb{R}^2; 0 \leq x \leq 1, 0 \leq y \leq 1\}$. We require $\int_S |f(x, y)| dx dy < \infty$.

Note that $\int_S |f(x,y)|dxdy \geq \int_A |f(x,y)|dxdy$. Now if f were integrable over A, we could use Fubini's theorem to write it as repeated integral. But consider

$$\int_0^1 x \left(\int_0^1 \frac{y}{(x^2+y^2)^2} dy \right) dx = \frac{1}{2} \int_0^1 \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx.$$

Since $x \to \frac{1}{x}$ is not integrable over $[0,1]$, the result follows.