MAS 350/451/6352 Measure and Probability - Exercises

Problems for Chapter 1

1. Give a careful proof by induction of the fact that if $\mathbf{B}$ is a Boolean algebra and $A_1, A_2, \ldots, A_n \in \mathbf{B}$, then $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathbf{B}$.

2. Show that if S is a set containing n elements, then the power set $\mathcal{P}(S)$ contains 2^n elements. [Hint: How many subsets are there of size $r (1 \leq r \leq n)$? The binomial theorem may also be of some use.]

3. Let Σ_1 and Σ_2 be σ-algebras of subsets of a set S. Define

$$\Sigma_1 \cap \Sigma_2 = \{A \subseteq S; A \in \Sigma_1 \text{ and } A \in \Sigma_2\}.$$

Show that $\Sigma_1 \cap \Sigma_2$ is a σ-algebra. Define $\Sigma_1 \cup \Sigma_2 = \{A \subseteq S; A \in \Sigma_1 \text{ or } A \in \Sigma_2\}$, (where "or" is inclusive.) Why is $\Sigma_1 \cup \Sigma_2$ not in general a σ-algebra?

4. If (S, Σ, m) is a measure space, show that for all $A, B \in \Sigma$

   (a) $m(A \cup B) + m(A \cap B) = m(A) + m(B)$,

   (b) $m(A \cup B) \leq m(A) + m(B)$.

Hence prove that if $A_1, A_2, \ldots, A_n \in \Sigma$,

$$m\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} m(A_i).$$

5. (a) If m is a measure on (S, Σ) and $k > 0$, show that km is also a measure on (S, Σ) where for all $A \in \Sigma$,

$$(km)(A) = km(A).$$

Hence show that if m is a finite measure then P is a probability measure where $P(A) = \dfrac{m(A)}{m(S)}$ for all $A \in \Sigma$. Let $[a, b]$ be a finite closed interval in $\mathbb{R}$. Write down a formula for the *uniform distribution* as a probability measure on $[a, b]$ using the above considerations and Lebesgue measure. [Hint: Recall that the uniform distribution has the property that subintervals of $[a, b]$ which have the same length, will have the same probability.]

(b) If m and n are measures on (S, Σ), deduce that $m+n$ is a measure on (S, Σ) where $(m+n)(A) = m(A) + n(A)$ for all $A \in \Sigma$.

6. If m is a measure on (S, Σ) and $B \in \Sigma$ is fixed, show that $m_B(A) = m(A \cap B)$ for $A \in \Sigma$ defines another measure on (S, Σ). If m is a finite measure and $m(B) > 0$, deduce that P_B is a probability measure where $P_B(A) = \dfrac{m_B(A)}{m(B)}$. How does this relate to the notion of conditional probability?

7. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite set and let $c_1, c_2, \ldots, c_n$ be non-negative numbers. Deduce that m is a measure on $(S, \mathcal{P}(S))$ where $m = \sum_{i=1}^{n} c_j \delta_{x_j}$. What condition should be imposed on $\{c_1, c_2, \ldots, c_n\}$ for m to be a probability measure?

8. Show that $\mathcal{B}(\mathbb{R})$ contains all closed intervals $[a, b](-\infty < a < b < \infty)$.

9. (a) Show that if m is a finite measure on (S, Σ) having total mass M then
$$m(A^c) = M - m(A).$$

   (b) Let m be a finite measure on (S, Σ). We say that a sequence of sets (A_n) with each $A_n \in \Sigma$ is *decreasing* to $A = \bigcap_{n=1}^{\infty} A_n$ if $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$. Show that $m(A) = \lim_{n \to \infty} m(A_n)$.
   [Hint: Use the result of (a).]

   (c) Use (b) to give a complete proof that the Cantor set has Lebesgue measure zero.

Problems for Chapter 2

10. Let (S, Σ) be a measurable space. Show that for all $A, B \in \Sigma$

   (a) $\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A \cap B}$,

   (b) $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$,

   (c) $\mathbf{1}_{A-B} = \mathbf{1}_A - \mathbf{1}_B$, if $B \subseteq A$,

   (d) $\mathbf{1}_{A \cap B} = \mathbf{1}_A \mathbf{1}_B$.

   Furthermore if (A_n) is a sequence of disjoint sets in Σ and $A = \bigcup_{n=1}^{\infty} A_n$, show that $\mathbf{1}_A = \sum_{n=1}^{\infty} \mathbf{1}_{A_n}$.

   [Hint: In (a) consider what happens to both sides in each of the four cases: $x \in A, x \in B; x \in A, x \notin B$, etc.]

11. Let (a_n) and (b_n) be bounded sequences of real numbers. Show that

    (a) $\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty}(-a_n)$,

    (b) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$,

    (c) $\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$,

    (d) if $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$, $\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$,

    (e) if $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$, $\liminf_{n\to\infty}(a_n b_n) \geq (\liminf_{n\to\infty} a_n)(\liminf_{n\to\infty} b_n)$,

    (f) $\limsup_{n\to\infty} |a_n| = 0 \Rightarrow (a_n)$ converges to 0.

12. Let (S, Σ) be a measurable space and $f : S \to \mathbb{R}$ be a constant function, i.e. there exists $c \in \mathbb{R}$ so that $f(x) = c$ for all $x \in S$. Show that f is measurable.

13. If (S, Σ) be a measurable space and $f : S \to \mathbb{R}$ show that f is measurable if and only if $f^{-1}((a, b)) \in \Sigma$ for all $-\infty \leq a < b \leq \infty$.

14. Let (S, Σ) be a measurable space and $f : S \to \mathbb{R}$ be a measurable function.

    (a) Show that $g = f + c$ is measurable, where $c \in \mathbb{R}$ is fixed,

    (b) Show that $g = kf$ is measurable, where $k \in \mathbb{R}$ is fixed.

15. Let (S, Σ) be a measurable space and $f : S \to \mathbb{R}$ be a measurable function. If $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable, show that $g \circ f$ is measurable from S to $\mathbb{R}$. What does this result tell us about functions of random variables in probability theory?

16. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Show that the mapping $h : \mathbb{R} \to \mathbb{R}$ is measurable, where $h(x) = f(x + y)$ for all $x \in \mathbb{R}$, and where $y \in \mathbb{R}$ is fixed.

17. Let (S, Σ) be a measurable space and $f : S \to \mathbb{R}$ be a function. Define the function $|f| : S \to \mathbb{R}$ by $|f|(x) = |f(x)|$ for all $x \in \mathbb{R}$. Show that

    (a) $f = f_+ - f_-$,

    (b) $|f| = f_+ + f_-$,

    (c) if f is measurable then $|f|$ is also measurable.

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Explain why both f and its derivative f' are measurable functions.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonic increasing. Show that it is measurable.

20. Let (S, Σ) be a measure space and (f_n) be a sequence of measurable functions from S to $\mathbb{R}$. Let $f : S \to \mathbb{R}$ be measurable. We say that $f_n \to f$ *almost everywhere* or *(a.e.)* for short, if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S - A$ where $m(A) = 0$. Show that if $f_n \to f$ (a.e.) and $g_n \to g$ (a.e.) then

    (a) $f_n^2 \to f^2$ (a.e.)

    (b) $f_n + g_n \to f + g$ (a.e.)

    (c) $f_n g_n \to fg$ (a.e.)

21. * A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *upper-semicontinuous* at $x \in \mathbb{R}$, if given any $\epsilon > 0$ there exists $\delta > 0$ so that $f(y) < f(x) + \epsilon$ whenever $|x - y| < \delta$.

    (a) Deduce that $f = \mathbf{1}_{[a,\infty)}$ (where $a \in \mathbb{R}$) is upper-semicontinuous for all $x \in \mathbb{R}$,

    (b) Deduce that the *floor function* $f(x) = \llcorner x \lrcorner$, which delivers the greatest integer less than or equal to x, is upper-semicontinuous for all $x \in \mathbb{R}$,

    (c) Show that if f is upper-semicontinuous for all $x \in \mathbb{R}$ then it is measurable.

Problems for Chapter 3

22. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$f = \begin{cases} 0 & \text{if } x < -2 \\ 1, & \text{if } -2 \leq x < -1, \\ 0 & \text{if } -1 \leq x < 0, \\ 2 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } 1 \leq x < 2, \\ 0, & \text{if } x \geq 2 \end{cases}$$

    Write f explicitly as a simple function and calculate $\int_{\mathbb{R}} f(x)dx$.

23. Let (S, Σ, m) be a measure space, $A \in \Sigma$ and f be a real-valued simple function defined on S. Show that $f\mathbf{1}_A$ is also a simple function, which is non-negative if f is. If f is non-negative, what constraint can you impose to ensure that $I_A(f) = I(f\mathbf{1}_A)$ is finite?

24. Prove Theorem 3.3.1 (2) to (4).

25. Prove the following version of *Chebychev's inequality*. If $f : S \to \mathbb{R}$ is a measurable function and $c > 0$ then

$$m(\{x \in S; |f(x)| \geq c\}) \leq \frac{1}{c^2} \int_S f^2 dm.$$

Formulate and prove a similar inequality where c^2 is replaced by c^p for $p \geq 1$.

[Hint: Imitate the method of proof for Markov's inequality.]

26. Let X be a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Derive the "probabilist's" version of Chebychev's inequality:

$$P(|X - \mu| \geq c) \leq \frac{\mathrm{Var}(X)}{c^2},$$

where the variance $\mathrm{Var}(X) = \mathbb{E}((X - \mu)^2)$ and $\mathbb{E}(X) = \mu$ is assumed to be finite.

[Hint: Derive this as a special case of the previous result. It does not need to be proved from scratch.]

27. Extend Corollary 3.3.1 as follows. Show that if f is a real valued measurable function for which $\int_S |f|^p dm = 0$ for some $p \geq 1$ then $f = 0$ (a.e.)

28. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$f = \begin{cases} 0 & \text{if } x < -2 \\ -1, & \text{if } -2 \leq x < -1, \\ 1 & \text{if } -1 \leq x < 0, \\ -2 & \text{if } 0 \leq x < 1, \\ 3 & \text{if } 1 \leq x < 2, \\ 0, & \text{if } x \geq 2 \end{cases}$$

Write down f_+ and f_- and confirm that they are non-negative simple functions. Calculate $\int_{\mathbb{R}} f(x)dx$.

29. Give the proof of Theorem 3.5.1 (1) and (3). [Hint: For (1), consider the cases $c \geq 0, c = -1$ and $c < 0$ $(c \neq -1)$ separately.]

30. Show that if f and g are integrable functions then

    (a) $\left|\int_S f dm\right| \leq \int_S |f| dm$,

(b) (*The triangle inequality.*) $\int_S |f+g|dm \leq \int_S |f|dm + \int_S |g|dm$.

31. Show that $f = g$ (a.e.) defines an equivalence relation on the set of all real-valued measurable functions defined on (S, Σ, m).

32. Recall Dirichlet's jump function $\mathbf{1}_{\mathbb{Q}}$. Does it make sense to write down the Fourier coefficients $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathbf{1}_{\mathbb{Q}}(x)\cos(nx)dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathbf{1}_{\mathbb{Q}}(x)\sin(nx)dx$ as Lebesgue integrals? If so, what values do they have? Can you associate a Fourier series to $\mathbf{1}_{\mathbb{Q}}$? If so, (and if it is convergent) what does it converge to?

33. Let (S, Σ, m) be a measure space and (A_n) be a sequence of disjoint sets with $A_n \in \Sigma$ for each $n \in \mathbb{N}$. define $A = \bigcup_{n=1}^{\infty} A_n$. If $f: S \to \mathbb{R}$ is measurable show that $f\mathbf{1}_A$ is integrable if and only if $f\mathbf{1}_{A_n}$ is integrable for each $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\int_{A_n}|f|dm < \infty$.

    [Hint: Use the monotone convergence theorem.]

34. Prove the *reverse Fatou lemma*, i.e. if (f_n) is a sequence of non-negative measurable functions for which $f_n \leq f$ for all $n \in \mathbb{N}$ where f is integrable then
$$\limsup_{n\to\infty}\int_S f_n dm \leq \int_S \limsup_{n\to\infty} f_n dm.$$
    [Hint. Apply Fatou's lemma to $f - f_n$.]

35. Show that if $f: \mathbb{R} \to \mathbb{R}$ is integrable then so are the mappings $x \to \cos(\alpha x)f(x)$ and $x \to \sin(\beta x)f(x)$, where $\alpha, \beta \in \mathbb{R}$. Deduce that
$$\lim_{n\to\infty}\int_{\mathbb{R}}\cos(x/n)f(x)dx = \int_{\mathbb{R}} f(x)dx.$$

36. *Parameter Dependent Integrals I.* Let (S, Σ, m) be a measure space and $f: [a,b] \times S \to \mathbb{R}$ be a measurable function for which

    (i) The mapping $x \to f(t,x)$ is integrable for all $t \in [a,b]$,

    (ii) The mapping $t \to f(t,x)$ is continuous for all $x \in S$,

    (iii) There exists a non-negative integrable function $g: S \to \mathbb{R}$ so that $|f(t,x)| \leq g(x)$ for all $t \in [a,b], x \in S$.

    Use the dominated convergence theorem to show that the mapping $t \to \int_S f(t,x)dm(x)$ is continuous on $[a,b]$.

    [Hint. Show that $\lim_{n\to\infty}\int_S f(t_n,x)dm(x) = \int_S f(t,x)dm(x)$ for any sequence (t_n) satisfying $\lim_{n\to\infty} t_n = t$.]

37. *Parameter Dependent Integrals II.* Let (S, Σ, m) be a measure space and $f : [a, b] \times S \to \mathbb{R}$ be a measurable function for which

   (i) The mapping $x \to f(t, x)$ is integrable for all $t \in [a, b]$,

   (ii)$'$ The mapping $t \to f(t, x)$ is differentiable for all $x \in S$,

   (iii)$'$ There exists a non-negative integrable function $h : S \to \mathbb{R}$ so that $\left|\dfrac{\partial f(t,x)}{\partial t}\right| \leq h(x)$ for all $t \in [a, b], x \in S$.

   Show that the mapping $t \to \int_S f(t, x)dm(x)$ is differentiable on (a, b) and that

$$\frac{d}{dt}\int_S f(t,x)dm(x) = \int_S \frac{\partial f(t,x)}{\partial t}dm(x).$$

   [Hint: Use the mean value theorem.]

38. Let (S, Σ, m) be a measure space. We say that a sequence (f_n) of integrable functions from S to $\mathbb{R}$ *converges in* $\mathcal{L}_1$ to an integrable function f if $\lim_{n\to\infty} \int_S |f_n - f|dm = 0$. Show that if (g_n) also converges in $\mathcal{L}_1$ to g then $(f_n + g_n)$ converges to $f + g$.

39. Consider the sequence (f_n) on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ where $f_n = n\mathbf{1}_{(0,1/n)}$. Show that (f_n) converges pointwise to zero, but that no subsequence of (f_n) converges to zero in $\mathcal{L}_1$.

40. Deduce that $f(x) = \lim_{n\to\infty} f_n(x)$ where $f(x) = -2xe^{-x^2}$ and $f_n(x) = \sum_{r=1}^{n}[-2r^2xe^{-r^2x^2} + 2(r+1)^2xe^{-(r+1)^2x^2}]$ for $x \in \mathbb{R}$. Show that f and f_n are (Riemann) integrable over $[0, a]$ for all $n \in \mathbb{N}$ but that $\int_0^a f(x)dx \neq \lim_{n\to\infty}\int_0^a f_n(x)dx$. (This example is due to Darboux.)

Problems for Chapter 4

41. Write down probabilistic versions of the monotone convergence theorem, Fatou's lemma and the dominated convergence theorem, using random variables in place of measurable functions and expectation in place of the integral.

42. Let $(\Omega, \mathcal{F}, P)$ be a probability space and (A_n) a sequence of sets in $\mathcal{F}$. We have defined independence of (A_n) in terms of finite subsets, but a more obvious approach might appear to be the requirement that

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} P(A_n),$$

where the right hand side is the limit $\lim_{n\to\infty} P(A_1)P(A_2)\cdots P(A_n)$. Why is this not a sensible idea?

43. Establish Theorem 4.2.2 (6)

44. Let X and Y be independent random variables and $f, g : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Deduce that $f(X)$ and $g(Y)$ are also independent.

45. Let X be a random variable and $a \in \mathbb{R}$. Prove that

$$\mathbb{E}(\max\{X, a\}) \geq \max\{\mathbb{E}(X), a\}.$$

[Hint: Write $\mathbb{E}(\max\{X, a\})$ as an integral.]

46. Let X be a random variable that takes positive integer values.

    (a) Deduce that $X = \sum_{i=1}^{\infty} \mathbf{1}_{\{X \geq i\}}$.

    (b) Show that $\mathbb{E}(X) = \sum_{i=1}^{\infty} P(X \geq i)$,

47. Let X be a random variable that takes non-negative values only. Show that

$$\sum_{i=1}^{\infty}(i-1)\mathbf{1}_{A_i} \leq X < \sum_{i=1}^{\infty} i\mathbf{1}_{A_i},$$

where $A_i = \{i - 1 \leq X < i\}$ for $i \in \mathbb{N}$. Hence deduce that

$$\sum_{k=1}^{\infty} P(X \geq k) \leq \mathbb{E}(X) < 1 + \sum_{k=1}^{\infty} P(X \geq k).$$

48. Let Ω be a set and $\mathcal{F}$ be a σ-algebra of subsets of Ω. If (A_n) is a sequence of sets in $\mathcal{F}$ confirm that $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.

49. (a) If (A_n) is as in the last question, show that

$$\left(\limsup_{n\to\infty} A_n\right)^c = \liminf_{n\to\infty} A_n^c,$$

and hence deduce that

$$P\left(\limsup_{n\to\infty} A_n\right) = 1 - P\left(\liminf_{n\to\infty} A_n^c\right).$$

    (b) Use (a) to establish the last part of Theorem 4.3.1, i.e. show that $\limsup_{n\to\infty} P(A_n) \leq P\left(\limsup_{n\to\infty} A_n\right)$. (Note that you can also prove this directly by using continuity of probability, as was done for the left hand part of the inequality.)

50. Prove that in a sequence of independent coin tosses, runs of heads or tails of arbitrary length can occur infinitely often.

51. Let X be a real-valued random variable with law p_X defined on a probability space $(\Omega, \mathcal{F}, P)$. Show that for all bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$,
$$\int_\Omega f(X(\omega))dP(\omega) = \int_\mathbb{R} f(x)dp_X(x).$$
What can you say about these integrals when f is non-negative but not necessarily bounded?

    [Hint: For the first part, begin with f an indicator function, then extend to simple, bounded non-negative and general bounded measurable functions.]

52. Suppose that X and Y are random variables wherein both X^2 and Y^2 are integrable. Prove the *Cauchy-Schwarz inequality*:
$$|\mathbb{E}(XY)| \leq (\mathbb{E}(X^2)^{\frac{1}{2}})(\mathbb{E}(Y^2)^{\frac{1}{2}}).$$

    [Hint: Consider $g(t) = \mathbb{E}((X + tY)^2)$ as a quadratic function of $t \in \mathbb{R}$.]

    Deduce that if X^2 is integrable, then so is X and that $|\mathbb{E}(X)|^2 \leq \mathbb{E}(X^2)$.

53. Deduce that for a random variable X having a finite mean μ, $\mathbb{E}(X^2) < \infty$ if and only if $\mathrm{Var}(X) < \infty$. Show further that $\mathbb{E}(|X - \mu|)^2 \leq \mathrm{Var}(X)$.

54. Let X be a random variable for which $\mathbb{E}(|X|^n) < \infty$. Show that $\mathbb{E}(|X|^m) < \infty$ for all $1 \leq m < n$.

    [Hint: Write $X = X\mathbf{1}_{\{X \leq 1\}} + X\mathbf{1}_{\{X > 1\}}$.]

55. Show that the converse to Theorem 4.4.1 (2) is false by taking $\Omega = [0, 1], \mathcal{F} = \mathcal{B}([0, 1])$ and P to be Lebesgue measure. Take $X = 0$ and define $X_n = \mathbf{1}_{A_n}$ where $A_1 = [0, 1/2], A_2 = [1/2, 1], A_3 = [0, 1/4], A_4 = [1/4, 1/2], A_5 = [1/2, 3/4], A_6 = [3/4, 1], A_7 = [0, 1/8], A_8 = [1/8, 1/4]$ etc.

56. Let (S, Σ) be a measurable space and $f : S \to \mathbb{C}$ be a (complex-valued) measurable function. Deduce that f is integrable if and only if $|f|$ is.

57. If X is a normally distributed random variable with mean μ and variance σ^2, deduce that its characteristic ϕ_X function is given for each $u \in \mathbb{R}$ by $\phi_X(u) = \exp\left\{i\mu u - \frac{1}{2}\sigma^2 u^2\right\}$.

[Hint: First show that it is sufficient to establish the case $\mu = 0$ and $\sigma = 1$ by writing $Y = \frac{1}{\sigma}(X - \mu)$. Then show that $y \to \phi_Y(u)$ is differentiable and deduce that $\phi_Y'(u) = -u\phi_Y(u)$. Now solve the initial value problem using what you know about $\phi_Y(0)$.]

58. Suppose that X is a random variable for which $\mathbb{E}(|X|^n) < \infty$ for some n. Explain carefully why

$$\mathbb{E}(X^n) = i^{-n} \left. \frac{d^n}{du^n}\phi_X(u) \right|_{u=0}.$$

59. (a) Let X be a non-negative random variable and $a > 0$. Show that $\mathbb{E}(e^{-aX}) \leq 1$.

    (b) A random variable is said to have an *exponential moment* if $\mathbb{E}(e^{a|X}) < \infty$ for some $a > 0$. Show that if $X \sim N(0,1)$ then it has exponential moments for all $a > 0$.

    (c) If X has an exponential moment, deduce that it has moments to all orders, i.e. that $\mathbb{E}(|X|^n) < \infty$ for all $n \in \mathbb{N}$.

60. Show that the conclusion of the weak law of large numbers continue to hold if the requirement that the random variables (X_n) are i.i.d. is replaced by the weaker condition that they are identically distributed, and *uncorrelated*, i.e. $\mathbb{E}(X_m X_n) = \mathbb{E}(X_m)\mathbb{E}(X_n)$ whenever $m \neq n$.

61. The first central limit theorem (CLT) to be established was due to de Moivre and Laplace. In this case each X_n takes only two values, 1 with probability p and 0 with probability $1 - p$ where $0 < p < 1$ (i.e. the X_ns are i.i.d. *Bernoulli* random variables.) Write down the form of the CLT in this case (writing the standardised random variable Y_n in terms of $S_n = X_1 + X_2 + \cdots + X_n$), and explain its relation to the "binomial approximation to the normal distribution".