MAS451/6352 Measure and Probability - Additional Exercises on Outer Measures

1. Define functions μ_1^* to μ_6^* on $\mathcal{P}(\mathbb{R})$ by

$$\mu_1^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset \end{cases}$$

$$\mu_2^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases}$$

$$\mu_3^*(A) = \begin{cases} 0 & \text{if } A \text{ is bounded} \\ 1 & \text{if } A \text{ is unbounded} \end{cases}$$

$$\mu_4^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \text{ is non-empty and bounded} \\ \infty & \text{if } A \text{ is unbounded} \end{cases}$$

$$\mu_5^*(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is uncountable} \end{cases}$$

$$\mu_6^*(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ \infty & \text{if } A \text{ is uncountable} \end{cases}$$

   (a) Which of the above set functions are outer measures?

   (b) For each i such that μ_i^* is an outer measure, determine the μ_i^*-measurable subsets of $\mathbb{R}$.

2. Let C be a countable subset of $\mathbb{R}$. Using only the definition of λ^*, show that $\lambda^*(C) = 0$.

3. Let S be a set, $\mathcal{A}$ be a Boolean algebra of subsets of S and μ be a finitely additive measure on $(S, \mathcal{A})$. For each $A \subseteq S$, define

$$\mu^*(A) = \inf \sum_{n=1}^{\infty} \mu(A_n),$$

where the inf is taken over all sequences of subsets of S, $(A_n, n \in \mathbb{N})$ for which $A \subseteq \bigcup_{n=1}^{\infty} A_n$.

   (a) Show that μ^* is an outer measure on S.

   (b) Show that each set in $\mathcal{A}$ is μ^*-measurable.

(c) Show that if μ is countably additive, then $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$.

(d) Deduce that if μ is countably additive, then there exists a measure on $(S, \sigma(\mathcal{A}))$ which agrees with μ on $(S, \mathcal{A})$.

4. Let $c \in \mathbb{R}$ and for all $A \in \mathcal{P}(\mathbb{R})$ define the set,

$$A + c = \{x + c; x \in A\}.$$

(a) Show that if $A = (a, b)$ then $A + c = (a + c, b + c)$.

(b) Deduce that if $A = \bigcup_{n=1}^{\infty}(a_n, b_n)$ then $A + c = \bigcup_{n=1}^{\infty}(a_n + c, b_n + c)$.

(c) (*Translation Invariance of Lebesgue Outer Measure.*) Show that $\lambda^*(A + c) = \lambda^*(A)$, for all $A \subseteq \mathbb{R}$.

(d) (*Translation Invariance of Lebesgue Measure.*) If $A \in \mathcal{B}(\mathbb{R})$, show that $A + c \in \mathcal{B}(\mathbb{R})$, and hence deduce that $\lambda(A + c) = \lambda(A)$, for all $A \in \mathcal{B}(\mathbb{R})$.

[Hint: Its probably best to do part (d) after Chapter 2. You may also find it useful to introduce the *translation mapping* $\tau_c : \mathbb{R} \to \mathbb{R}$ defined by $\tau_c(x) = x - c$, for all $x \in \mathbb{R}$. Is τ_c measurable?]