MAS451/6352 Measure and Probability - Additional Exercises on Product Measures and Fubini's Theorem

Throughout these problems (S_1, Σ_1, m_1) and (S_2, Σ_2, m_2) are measure spaces. From Problem 3 onwards they are always σ-finite.

1. If $E, F \subset S_1 \times S_2$ and $x \in S_1$, show that

   (a) $(E \cap F)_x = E_x \cap F_x$,

   (b) $(E^c)_x = (E_x)^c$,

   (c) $\left(\bigcup_{n=1}^{\infty} E_n\right)_x = \bigcup_{n=1}^{\infty} (E_n)_x$, where (E_n) is a sequence of subsets of $S_1 \times S_2$.

2. If m_1 and m_2 are σ-finite measures, show that the product measure $m_1 \times m_2$ is also σ-finite.

3. If $A \in \Sigma_1$ and $B \in \Sigma_2$, prove that $(m_1 \times m_2)(A \times B) = m_1(A)m_2(B)$.

4. A product set $A_1 \times A_2$ is said to be *finite* if $m_i(A_i) < \infty$ for $i = 1, 2$. Show that product measure $m_1 \times m_2$ is the *unique* measure μ on $(S_1 \times S_2, \Sigma_1 \otimes \Sigma_2)$ for which

$$\mu(A_1 \times A_2) = m_1(A_1)m_2(A_2),$$

   for all finite product sets.

   [Hint: Use Dynkin's $\pi - \lambda$ theorem.]

5. (a) Let $f : S_1 \to \mathbb{R}$ and $g : S_2 \to \mathbb{R}$ be measurable functions. Define $h : S_1 \times S_2 \to \mathbb{R}$ by

$$h(x, y) = f(x)g(y),$$

   for all $x \in S_1, y \in S_2$. Show that h is measurable.

   (b) If f and g are integrable, show that h is also integrable and that

$$\int_{S_1 \times S_2} h \, d(m_1 \times m_2) = \left(\int_{S_1} f dm_1\right)\left(\int_{S_2} g dm_2\right).$$

6. Let $a_{ij} \geq 0$ for all $1 \leq i, j \leq \infty$. Show that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij},$$

   (in the sense that both double series converge to the same limit, or diverge together).

7. Let (S, Σ, m) be a σ-finite measure space and $f : S \to \mathbb{R}$ be a non-negative measurable function. Define $A_f = \{(x, t) \in S \times \mathbb{R}; 0 \le t \le f(x)\}$. Show that $A_f \in \Sigma \otimes \mathcal{B}(\mathbb{R})$ and that

$$(m \times \lambda)(A_f) = \int_S f dm.$$

8. Use Fubini's theorem to prove that

$$\lim_{T \to \infty} \int_0^T \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

[Hint: Write $\frac{1}{x} = \int_0^\infty e^{-xy} dy$.]

9. (a) Show that for the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = \frac{xy}{(x^2 + y^2)^2}$, the iterated integrals $\int_{-1}^1 \left(\int_{-1}^1 f(x, y) dy \right) dx$ and $\int_{-1}^1 \left(\int_{-1}^1 f(x, y) dx \right) dy$ exist and are equal.

   (b) Show that f is not integrable over the square $-1 \le x \le 1$ and $-1 \le y \le 1$.