MAS350 Measure and Probability- Solutions to Set Problems 1

5. (a) To show that km is a measure first observe that $m(A) \geq 0 \Rightarrow (km)(A) = km(A) \geq 0$ for all $A \in \Sigma$. We must establish M(i) and M(ii).

   M(i) $(km)(\emptyset) = k.m(\emptyset) = k.0 = 0$.

   M(ii) If (A_n) is a sequence of disjoint sets in Σ,

$$(km)\left(\bigcup_{n=1}^{\infty} A_n\right) = k.m\left(\bigcup_{n=1}^{\infty} A_n\right) = k\left(\sum_{n=1}^{\infty} m(A_n)\right) = \sum_{n=1}^{\infty}(km)(A_n).$$

   P is a probability measure by the result just proved where we take $k = 1/m(S)$.

   The uniform distribution on $[a, b]$ is obtained by taking m to be Lebesgue measure so if $A \in \mathcal{B}([a, b])$,

$$P(A) = \frac{\lambda(A)}{\lambda([a, b])} = \frac{\lambda(A)}{b - a}.$$

   For example if $A = [c, d]$, where $a \leq c < d \leq b$, then $P(A) = \frac{d-c}{b-a}$.

   (b) For all $A \in \Sigma, (m + n)(A) = m(A) + n(A) \geq 0$.

   M(i) $(m + n)(\emptyset) = m(\emptyset) + n(\emptyset) = 0 + 0 = 0$.

   M(ii) If (A_k) is a sequence of disjoint sets in Σ,

$$\begin{aligned}(m+n)\left(\bigcup_{k=1}^{\infty} A_k\right) &= m\left(\bigcup_{k=1}^{\infty} A_k\right) + n\left(\bigcup_{k=1}^{\infty} A_k\right) \\ &= \sum_{k=1}^{\infty} m(A_k) + \sum_{k=1}^{\infty} n(A_k) = \sum_{k=1}^{\infty}(m+n)(A_k).\end{aligned}$$

6. To show that m_B is a measure, first note that $m(A) \geq 0 \Rightarrow m(A \cap B) = m_B(A) \geq 0$ for all $A \in \Sigma$.

   M(i) $m_B(\emptyset) = m(B \cap \emptyset) = m(\emptyset) = 0$.

   M(ii) Note that if (A_n) is a sequence of mutually disjoint sets in Σ then so is $(A_n \cap B)$ since if $n \neq p, (A_n \cap B) \cap (A_p \cap B) = (A_n \cap A_p) \cap B = \emptyset$. We then get

$$m_B\left(\bigcup_{n=1}^{\infty} A_n\right) = m\left(B \cap \bigcup_{n=1}^{\infty} A_n\right)$$

$$
\begin{aligned}
&= m\left(\bigcup_{n=1}^{\infty}(A_n \cap B)\right) \\
&= \sum_{n=1}^{\infty} m(A_n \cap B) = \sum_{n=1}^{\infty} m_B(A_n).
\end{aligned}
$$

P_B is a probability measure since

$$P_B(S) = \frac{m_B(S)}{m(B)} = \frac{m(S \cap B)}{m(B)} = \frac{m(B)}{m(B)} = 1.$$

If m is a probability measure and $A \in \Sigma$ then $P_B(A)$ is precisely the conditional probability of the event A given B, which is usually written $P(A|B)$.

9. (a) Since $A \cup A^c = S$ and $A \cap A^c = \emptyset$, we have

$$M = m(S) = m(A) + m(A^c).$$

(b) If $A = \bigcap_{n=1}^{\infty} A_n$, $A^c = \bigcup_{n=1}^{\infty} A_n^c$. (A_n^c) is increasing since $A_{n+1} \subseteq A_n \Rightarrow A_n^c \subseteq A_{n+1}^c$. Now using (a), Theorem 1.5.1 and (1.3.2) we find that

$$
\begin{aligned}
m(A) &= M - m(A^c) \\
&= M - \lim_{n\to\infty} m(A_n^c) \\
&= M - \lim_{n\to\infty} m(S - A_n) \\
&= M - \lim_{n\to\infty} (m(S) - m(A_n)) \\
&= \lim_{n\to\infty} m(A_n).
\end{aligned}
$$

(c) Since the Cantor set $C = \bigcap_{n=1}^{\infty} C_n$ where $C_{n+1} \subseteq C_n$, we can apply (b) to the calculations made in the notes to deduce that

$$
\begin{aligned}
\lambda(C) &= \lim_{n\to\infty} \lambda(C_n) \\
&= 1 - \lim_{n\to\infty} \sum_{r=1}^{n} \frac{2^{r-1}}{3^r} \\
&= 1 - \sum_{r=1}^{\infty} \frac{2^{r-1}}{3^r} = 0.
\end{aligned}
$$

Alternatively, $m(C_1) = \frac{2}{3}, m(C_2) = \frac{4}{9}$ and iterating yields $m(C_n) = \left(\frac{2}{3}\right)^n$ for all n and so, by (b)

$$m(C) = \lim_{n\to\infty} m(C_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$