MAS350 Measure and Probability- Solutions to Set Problems 2

13 Suppose that $f^{-1}((a,b)) \in \Sigma$ for all $-\infty \leq a < b \leq \infty$. Then in particular, when we take $b = \infty$ we find that $f^{-1}((a,\infty)) \in \Sigma$ for all $a \in \mathbb{R}$ and so f is measurable.

Conversely if f is measurable then for all $a, b \in \mathbb{R}$, $f^{-1}((a,\infty)) \in \Sigma$ and $f^{-1}((-\infty,b)) \in \Sigma$ by Theorem 2.2.1 (c). So

$$\begin{aligned} f^{-1}((a,b)) &= f^{-1}((-\infty,b) \cap (a,\infty)) \\ &= f^{-1}((-\infty,b)) \cap f^{-1}((a,\infty)) \in \Sigma, \end{aligned}$$

since Σ is closed under intersections. A similar argument covers the cases $a = -\infty, b \in \mathbb{R}$ and $a \in \mathbb{R}, b = \infty$. For the case $a = -\infty, b = \infty$, we have $f^{-1}((-\infty,\infty))) = f^{-1}(\mathbb{R}) = S \in \Sigma$.

14 (a)

$$\begin{aligned} g^{-1}((a,\infty)) &= \{x \in S; g(x) \in (a,\infty)\} \\ &= \{x \in S; f(x) + c \in (a,\infty)\} \\ &= \{x \in S; f(x) \in (a-c,\infty)\} \\ &= f^{-1}((a-c,\infty)) \in \Sigma, \end{aligned}$$

as f is measurable. Hence g is measurable.

(b) If $k = 0, g = 0 = \mathbf{1}_{\emptyset}$ and so is measurable, as it is the indicator function of a measurable set.
If $k > 0, g^{-1}((a,\infty)) = f^{-1}((a/k,\infty)) \in \Sigma$ since f is measurable, and so g is measurable.
If $k < 0$, since $f(x) < a/k$ if and only if $kf(x) > a$, we have $g^{-1}((a,\infty)) = f^{-1}((-\infty,a/k)) \in \Sigma$ by Theorem 2.2.1 (iii), and so g is measurable.

An alternative (and quicker) approach to both (a) and (b), is to use Theorem 2.3.2, taking $G(y) = y + c$ in (a), and $G(y) = ky$ in (b), for all $y \in \mathbb{R}$.

17 (a) For each $x \in \mathbb{R}$, either $f(x) \geq 0$ or $f(x) < 0$. If $f(x) \geq 0$ then $f_-(x) = 0$ and $f(x) = f_+(x) = f_+(x) - f_-(x)$ as required. If $f(x) < 0$ then $f_+(x) = 0$ and $f(x) = -f_-(x)$ and we again have $f(x) = f_+(x) - f_-(x)$.

(b) As in (a), if $f(x) \geq 0$ then $|f(x)| = f(x) = f_+(x) = f_+(x) + f_-(x)$. If $f(x) < 0, |f(x)| = -f(x) = f_-(x) = f_+(x) + f_-(x)$.

(c) If f is measurable then both f_+ and f_- are measurable by Corollary 2.3.2, and then $|f|$ is measurable by (b) and Theorem 2.3.1.