MAS350 Measure and Probability- Solutions to Set Problems 3

24 (2) By Theorem 3.2.1 we know this holds for simple functions. Since any simple function s can be written $s = \alpha t$ where t is simple, we have

$$\begin{aligned} I(\alpha f) &= \sup\left\{\int_S sdm, s \text{ simple}, 0 \leq s \leq \alpha f\right\} \\ &= \sup\left\{\int_S \alpha tdm, t \text{ simple}, 0 \leq \alpha t \leq \alpha f\right\} \\ &= \sup\left\{\alpha \int_S tdm, t \text{ simple}, 0 \leq t \leq f\right\} \\ &= \alpha \sup\left\{\int_S tdm, t \text{ simple}, 0 \leq t \leq f\right\} = \alpha I(f). \end{aligned}$$

(3) This follows from (1) (monotonicity) since $f\mathbf{1}_A \leq f\mathbf{1}_B$.

(4)

$$I_A(f) = \sup\left\{\int_S sdm, s \text{ simple}, 0 \leq s \leq f\mathbf{1}_A\right\}.$$

Since $f\mathbf{1}_A(x) = 0$ when $x \in A^c$, $s \leq f\mathbf{1}_A$ if and only if $s\mathbf{1}_A \leq f\mathbf{1}_A$ and so

$$I_A(f) = \sup\left\{\int_S s\mathbf{1}_A dm, s \text{ simple}, 0 \leq s\mathbf{1}_A \leq f\mathbf{1}_A\right\} = 0,$$

since $\int_S s\mathbf{1}_A dm = 0$ for all simple s by (3.2.2).

30 (a) Since $\int_S f dm = \int_S f_+ dm - \int_S f_- dm$ we have,

$$\begin{aligned} \left|\int_S f dm\right| &= \max\left\{\int_S f_+ dm - \int_S f_- dm, \int_S f_- dm - \int_S f_+ dm\right\} \\ &\leq \int_S f_+ dm + \int_S f_- dm \\ &= \int_S (f_+ + f_-) dm \text{ by Theorem 3.4.2} \\ &= \int_S |f| dm. \end{aligned}$$

Alternatively, you could start the solution by using the triangle inequality for real numbers to write:

$$\left|\int_S f dm\right| = \left|\int_S f_+ dm - \int_S f_- dm\right| \leq \int_S f_+ dm + \int_S f_- dm,$$

and then continue as in the last two lines above.

(b) Since $|f(x)+g(x)| \leq |f(x)|+|g(x)|$ for all $x \in S$, using monotonicity and Theorem 3.4.2:

$$\int_S |f+g|dm \leq \int_S (|f|+|g|)dm = \int_S |f|dm + \int_S |g|dm.$$

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$$\frac{\partial}{\partial t}\int_S f(t,x)dm(x) = \lim_{c\to 0}\int_S \frac{f(t+c,x)-f(t,x)}{c}dm(x).$$

By the mean value theorem, for each $x \in S$ there exists $0 < \theta(x) < 1$ so that $\left|\frac{f(t+c,x)-f(t,x)}{c}\right| = \frac{\partial f}{\partial t}(t+\theta(x)c,x)$ and by assumption (iii)$'$, $\left|\frac{\partial f}{\partial t}(t+\theta(x)c,x)\right| \leq h(x)$, where h is integrable. So using the dominated convergence theorem (where we implicitly replace c by an arbitrary sequence (c_n) as in the solution to Problem 36), we obtain

$$\begin{aligned}\lim_{c\to 0}\int_S \frac{f(t+c,x)-f(t,x)}{c}dm(x) &= \int_S \lim_{c\to 0}\frac{f(t+c,x)-f(t,x)}{c}dm(x)\\ &= \int_S \frac{\partial f(t,x)}{\partial t}dm(x).\end{aligned}$$