MAS 350/451/6352 Measure and Probability: Solutions to Week 10 Problems, Chapter 3 Problems 33, 34, 35, 36; Chapter 4 Problems 41, 43 and also all other solutions that have not yet been posted, up to and including 42.

29 (1) Using Theorem 3.3.1 (2), if $c \geq 0$,

$$\int_S cf dm = \int_S cf_+ dm - \int_S cf_- dm = c\int_S f_+ dm - c\int_S f_- dm = c\int_S f dm.$$

If $c = -1, (-f)_+ = f_-$ and $(-f)_- = f_+$ and so

$$\int_S (-f) dm = \int_S f_- dm - \int_S f_+ dm = -\left(\int_S f_+ dm - \int_S f_- dm\right) = -\int_S f dm.$$

Finally if $c < 0 (c \neq -1)$ write $c = -d$ where $d > 0$ and use the two cases we've just proved.

(3) If $f \leq g$ then $g - f \geq 0$ so by Theorem 3.3.1 (1), $\int_S (g - f) dm \geq 0$. But by (1) and (2) this is equivalent to $\int_S g dm - \int_S f dm \geq 0$, i.e. $\int_S g dm \geq \int_S f dm$, as required.

31 Reflexivity is obvious as $f(x) = f(x)$ for all $x \in S$. So is symmetry as $f = g$ (a.e.) if and only if $g = f$ (a.e.). For transitivity, let $A = \{x \in S; f(x) \neq g(x)\}, B = \{x \in S; g(x) \neq h(x)\}$ and $C = \{x \in S; f(x) \neq h(x)\}$. Then $C \subseteq A \cup B$ and so $m(C) \leq m(A) + m(B) = 0$.

32 $x \to \mathbf{1}_{\mathbb{Q}}(x)\cos(nx)$ is integrable as $|\mathbf{1}_{\mathbb{Q}}(x)\cos(nx)| \leq |\cos(nx)|$ for all $x \in \mathbb{R}$ and $x \to \cos(nx)$ is integrable. Similarly $x \to \mathbf{1}_{\mathbb{Q}}(x)\sin(nx)$ is integrable. So the Fourier coefficients a_n and b_n are well-defined as Lebesgue integrals. As $|\cos(nx)| \leq 1$, we have $a_n = 0$ for all $n \in \mathbb{Z}_+$ since,

$$\begin{aligned} |a_n| &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} \mathbf{1}_{\mathbb{Q}}(x)|\cos(nx)|dx \\ &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} \mathbf{1}_{\mathbb{Q}}(x)dx = 0. \end{aligned}$$

By a similar argument, $b_n = 0$ for all $n \in \mathbb{N}$. So it is possible to associate a Fourier series to $\mathbf{1}_{\mathbb{Q}}$ which converges pointwise to zero. This illustrates that pointwise convergence is not the right tool for examining convergence of Fourier series. Pointwise convergence a.e. (i.e. pointwise convergence everywhere except on a set of measure zero) is more appropriate.

33 First suppose that $f\mathbf{1}_A$ is integrable. Then for all $n \in \mathbb{N}, |f|\mathbf{1}_{A_n} \leq |f|\mathbf{1}_A$ and so $f\mathbf{1}_{A_n}$ is integrable by monotonicity. It follows that

$$\sum_{r=1}^{n} \int_S |f|\mathbf{1}_{A_r} dm = \int_S |f|\mathbf{1}_{\bigcup_{r=1}^n A_r} dm < \infty.$$

Now $|f|\mathbf{1}_{\bigcup_{r=1}^n A_r}$ increases to $|f|\mathbf{1}_A$ as $n \to \infty$ and so by the monotone convergence theorem,

$$\sum_{r=1}^{\infty} \int_S |f|\mathbf{1}_{A_r} dm = \lim_{n\to\infty} \int_S |f|\mathbf{1}_{\bigcup_{r=1}^n A_r} dm = \int_S |f|\mathbf{1}_A dm < \infty.$$

Conversely if $f\mathbf{1}_{A_n}$ is integrable for each $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \int_{A_n} |f| dm < \infty$, we have by Theorem 3.3.2 that

$$\begin{aligned} \int_S |f|\mathbf{1}_A dm &= \int_S |f|\mathbf{1}_{\bigcup_{n=1}^\infty A_n} dm \\ &= \sum_{n=1}^{\infty} \int_{A_n} |f| dm < \infty. \end{aligned}$$

34 $f - f_n \geq 0$ for all $n \in \mathbb{N}$ so by Fatou's lemma:

$$\liminf_{n\to\infty} \int_S (f - f_n) dm \geq \int_S \liminf_{n\to\infty} (f - f_n) dm.$$

$$\text{i.e. } \int_S f dm + \liminf_{n\to\infty} \int_S (-f_n) dm \geq \int_S f dm + \int_S \liminf_{n\to\infty} (-f_n) dm,$$

$$\text{and so } \liminf_{n\to\infty} - \left(\int_S f_n dm \right) \geq \int_S \liminf_{n\to\infty} (-f_n) dm.$$

Multiplying both sides by -1 reverses the inequality to yield

$$-\liminf_{n\to\infty} - \left(\int_S f_n dm \right) \leq \int_S \left(-\liminf_{n\to\infty} (-f_n) \right) dm.$$

But then by definition of $\limsup_{n\to\infty}$ we have

$$\limsup_{n\to\infty} \int_S f_n dm \leq \int_S \limsup_{n\to\infty} f_n dm.$$

35 Integrability follows easily from the facts that $|\cos(\alpha x)| \leq 1$ and $|\sin(\beta x)| \leq 1$ for all $x \in \mathbb{R}$. As $|\cos(x/n)f(x)| \leq |f(x)|$ for all $x \in \mathbb{R}$ and f is integrable, we may use the dominated convergence theorem to deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}} \cos(x/n)f(x)dx = \int_{\mathbb{R}} \lim_{n\to\infty} \cos(x/n)f(x)dx = \int_{\mathbb{R}} f(x)dx,$$

since $\lim_{n\to\infty}\cos(x/n) = \cos(0) = 1$ for all $x \in \mathbb{R}$.

36 Define $f_n(x) = f(t_n, x)$ for each $n \in \mathbb{N}, x \in S$. Then $|f_n(x)| \leq g(x)$ for all $x \in S$. Since g is integrable, by dominated convergence

$$\begin{aligned}\lim_{n\to\infty}\int_S f(t_n, x)dm(x) &= \int_S \lim_{n\to\infty} f_n(x)dm(x)\\ &= \int_S \lim_{n\to\infty} f(t_n, x)dm(x)\\ &= \int_S f(t, x)dm(x),\end{aligned}$$

where we used the continuity assumption (ii) in the last step.

38 Using the triangle inequality (Problem 29(b)),

$$\begin{aligned}\lim_{n\to\infty}\int_S |(f_n + g_n) - (f + g)|dm &= \lim_{n\to\infty}\int_S |(f_n - f) + (g_n - g)|dm\\ &\leq \lim_{n\to\infty}\int_S |(f_n - f|dm + \lim_{n\to\infty}\int_S |g_n - g)|dm\\ &\to 0,\end{aligned}$$

as $n \to \infty$.

39 Let $x \in \mathbb{R}$ be arbitrary. Then we can find $n_0 \in \mathbb{N}$ so that $\frac{1}{n_0} < |x|$ and then for all $n \geq n_0$, $f_n(x) = n\mathbf{1}_{(0,1/n)}(x) = 0$. So we have proved that $\lim_{n\to} f_n(x) = 0$. But for all $n \in \mathbb{N}$

$$\int_{\mathbb{R}} |f_n(x) - 0|dx = n\int_{\mathbb{R}} \mathbf{1}_{(0,1/n)}(x)dx = n.\frac{1}{n} = 1,$$

and so we cannot find any function in the sequence that gets arbitrarily close to 0 in the $\mathcal{L}_1$ sense.

40 For each $x \in \mathbb{R}, n \in \mathbb{N}$, the expression for $f_n(x)$ is a telescopic sum. If you begin to write it out, you see that terms cancel in pairs and you obtain

$$f_n(x) = -2xe^{-x^2} + 2(n + 1)^2 xe^{-(n+1)^2 x^2}.$$

Using the fact that $\lim_{N\to\infty} N^2 e^{-yN^2} = 0$, for all $y \in \mathbb{R}$ we find that

$$\lim_{n\to\infty} f_n(x) = f(x) = -2xe^{-x^2}.$$

The functions f and f_n are continuous and so Riemann integrable over the closed interval $[0, a]$. We can easily calculate $\int_0^a f(x)dx = -2\int_0^a xe^{-x^2}dx = e^{-a^2} - 1$. But on the other hand

$$\begin{aligned}\int_0^a f_n(x)dx &= \sum_{r=1}^{n} \int_0^a [-2r^2 xe^{-r^2x^2} + 2(r+1)^2 xe^{-(r+1)^2x^2}]dx \\ &= \sum_{r=1}^{n} (e^{-r^2 a} - e^{-(r+1)^2 a}) \\ &= e^{-a^2} - e^{-(n+1)^2 a} \to e^{-a^2} \text{ as } n \to \infty.\end{aligned}$$

So we conclude that $\int_0^a f(x)dx \neq \lim_{n\to\infty} \int_0^a f_n(x)dx$. This is an example where neither the monotone or dominated convergence theorems can be used (Why?) and illustrates that things can go badly wrong without them!

41 *Monotone Convergence Theorem.* Let (X_n) be an increasing sequence of non-negative random variables which converges pointwise to a random variable X, i.e. $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all $\omega \in \Omega$ (*). Then

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X).$$

Fatou's Lemma. Let (X_n) be a sequence of non-negative random variables, then

$$\liminf_{n\to\infty} \mathbb{E}(X_n) \leq \mathbb{E}\left(\liminf_{n\to\infty} X_n\right).$$

Dominated Convergence Theorem. Let (X_n) be a sequence of random variables which converges pointwise (*) to a random variable X. Suppose that there exists an integrable, non-negative random variable Y so that $|X_n(\omega)| \leq Y(\omega)$ for all $n \in \mathbb{N}$ and all $\omega \in \Omega$. Then X is integrable and

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X).$$

(*) We can in fact replace *pointwise convergence* by *convergence almost everywhere*, i.e. $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all $\omega \in \Omega - A$, where $A \in \mathcal{F}$ and $P(A) = 0$.

42 If $P(A_n) < 1$ for infinitely many ns, then $\prod_{n=1}^{\infty} P(A_n) = 0$ so $P\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} P(A_n)$ will hold if, say, two or more of the A_ns are disjoint, and that has nothing to do with independence.

43 Let (a_n) be a sequence diverging to ∞. We may without loss of generality assume that it is monotonic increasing. Define $A_n = \{\omega \in \Omega; X(\omega) \leq a_n\}$. Then (A_n) increases to Ω and by Theorem 4.2.1(a),

$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} P(A_n) = P(\Omega) = 1.$$

Next let $B_n = \{\omega \in \Omega; X(\omega) \leq -a_n\}$. Then (B_n) decreases to $\emptyset$ and by Theorem 4.2.1(b),

$$\lim_{x\to-\infty} F(x) = \lim_{n\to\infty} P(B_n) = P(\emptyset) = 0.$$