MAS 350/451/6352 Measure and Probability:
Solutions to Week 11 Problems, Chapter 4 Problems 44 to 55.

44 If $A, B \in \mathcal{B}(\mathbb{R})$ and f, g are Borel measurable, then $f^{-1}(A), g^{-1}(B) \in \mathcal{B}(\mathbb{R})$ and so

$$\begin{aligned} P(f(X) \in A, g(Y) \in B) &= P(X \in f^{-1}(A), Y \in g^{-1}(B)) \\ &= P(X \in f^{-1}(A))P(Y \in g^{-1}(B)) \\ &= P(f(X) \in A)P(g(Y) \in B). \end{aligned}$$

45 Using the usual notation, $a \vee b = \max\{a, b\}$,

$$\mathbb{E}(\max\{X, a\}) = \int_{\mathbb{R}} (x \vee a) dp_X(x).$$

Since $x \vee a \geq x$ and $x \vee a \geq a$, by monotonicity $\int_{\mathbb{R}} (x \vee a) dp_X(x) \geq \int_{\mathbb{R}} x dp_X(x) = \mathbb{E}(X)$ and $\int_{\mathbb{R}} (x \vee a) dp_X(x) \geq \int_{\mathbb{R}} a dp_X(x) = a$, and so

$$\mathbb{E}(\max\{X, a\}) \geq \max\{\mathbb{E}(X), a\}.$$

46 (a) Let $A_k = \{\omega \in \Omega; X(\omega) = k\}$. If $\omega \in A_k$, $X(\omega) = k$ and

$$\sum_{i=1}^{\infty} \mathbf{1}_{\{X \geq i\}}(\omega) = \mathbf{1}_{\{X \geq 1\}}(\omega) + \mathbf{1}_{\{X \geq 2\}}(\omega) + \cdots + \mathbf{1}_{\{X \geq k\}}(\omega) = k.$$

The result follows since we have the disjoint union $\Omega = \bigcup_{k=1}^{\infty} A_k$.

(b) By (a) and monotone convergence

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{\infty} \mathbf{1}_{\{X \geq i\}}\right) = \sum_{i=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{X \geq i\}}) = \sum_{i=1}^{\infty} P(X \geq i).$$

47 Let $\omega \in A_i$ then $i - 1 \leq X(\omega) < i$ and so

$$\sum_{i=1}^{\infty} (i-1)\mathbf{1}_{A_i}(\omega) \leq X(\omega) < \sum_{i=1}^{\infty} i\mathbf{1}_{A_i}(\omega),$$

from which the first result follows. For the second result, first observe that

$$\sum_{i=1}^{\infty} i\mathbf{1}_{A_i}(\omega) = \sum_{i=1}^{\infty} (i-1)\mathbf{1}_{A_i}(\omega) + \sum_{i=1}^{\infty} \mathbf{1}_{A_i}(\omega) = \sum_{i=1}^{\infty} (i-1)\mathbf{1}_{A_i}(\omega) + 1,$$

since $\sum_{i=1}^{\infty} \mathbf{1}_{A_i} = \mathbf{1}_{\bigcup_{i=1}^{\infty} A_i} = \mathbf{1}_\Omega = 1$. Arguing as in Problem 46(b), we find that

$$\sum_{k=1}^{\infty}(k-1)P(A_k) \leq \mathbb{E}(X) < 1 + \sum_{k=1}^{\infty}(k-1)P(A_k).$$

But $P(A_k) = P(k-1 \leq X < k)$ for each $k \in \mathbb{N}$ and

$$\begin{aligned}
\sum_{k=1}^{N}(k-1)P(k-1 \leq X < k) &= P(1 \leq X < 2) + 2P(2 \leq X < 3) \\
&+ 3P(3 \leq X < 4) + \cdots + (N-1)P(N-1 \leq X < N) \\
&= P(1 \leq X < N) + P(2 \leq X < N) \\
&+ P(3 \leq X < N) + \cdots + P(N-1 \leq X \leq N) \\
&\to \sum_{k=1}^{\infty} P(X \geq k) \text{ as } N \to \infty.
\end{aligned}$$

48 For any given $N \in \mathbb{N}$ and for each $m \geq N, \bigcap_{k \geq N} A_k \subseteq A_m$ and so $\bigcap_{k \geq N} A_k \subseteq \bigcup_{l \geq m} A_l$. Hence $\bigcap_{k \geq N} A_k \subseteq \bigcap_{m \geq N} \bigcup_{l \geq m} A_l$. Now $\omega \in \limsup_{n\to\infty} A_n$ if and only if ω is in infinitely many of the A_n's if and only if ω is in infinitely many of the A_n's for $n \geq N$. Hence $\limsup_{n\to\infty} A_n = \bigcap_{m \geq N} \bigcup_{l \geq m} A_l$, and we've proved that $\bigcap_{k \geq N} A_k \subseteq \limsup_{n\to\infty} A_n$. But since the choice of N was arbitrary, we have

$$\bigcup_{N \in \mathbb{N}} \bigcap_{k \geq N} A_k \subseteq \limsup_{n\to\infty} A_n,$$

i.e. $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$, as was required.

49 (a) Using de Morgan's laws

$$\left(\limsup_{n\to\infty} A_n\right)^c = \left(\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k\right)^c = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k^c = \liminf_{n\to\infty} A_n^c.$$

It follows that

$$P\left(\limsup_{n\to\infty} A_n\right) = 1 - P\left(\left(\limsup_{n\to\infty} A_n\right)^c\right) = 1 - P\left(\liminf_{n\to\infty} A_n^c\right).$$

(b) Using the part of Theorem 4.3.1 that we've already proved,

$$P\left(\limsup_{n\to\infty} A_n\right) = 1 - P\left(\liminf_{n\to\infty} A_n^c\right)$$

$$\begin{aligned} &\geq 1 - \liminf_{n\to\infty} P(A_n^c) \\ &= 1 - \liminf_{n\to\infty}(1 - P(A_n)) \\ &= -\liminf_{n\to\infty}\{-P(A_n)\} = \limsup_{n\to\infty} P(A_n). \end{aligned}$$

You can also obtain (b) directly by imitating the proof of the first part of Theorem 4.3.1 that is given in the notes.

50 We'll just deal with heads as the argument for tails is identical. Let E_m be the event that starting at the mth toss, n consecutive heads appear. Then $P(E_m) = 1/2^n$. The events $(E_m, E_{m+n}, E_{m+2n}, \ldots)$ are independent and $\sum_{r=1}^{\infty} P(E_{m+rn}) = \infty$. Then by the Borel-Cantelli lemma:

$$1 = P\left(\limsup_{r\to\infty} E_{m+rn}\right) \leq P\left(\limsup_{n\to\infty} E_n\right) = 1.$$

51 If f is an indicator function: $f = \mathbf{1}_A$ for some $A \in \mathcal{B}(\mathbb{R})$:

$$\int_\Omega \mathbf{1}_A(X(\omega))dP(\omega) = P(X \in A) = p_X(A) = \int_\mathbb{R} \mathbf{1}_A(x)p_X(dx),$$

and so the result holds in this case. It extends to simple functions by linearity. If f is non-negative and bounded

$$\begin{aligned} \int_\Omega f(X(\omega))dP(\omega) &= \sup\left\{\int_\Omega g(\omega)dP(\omega); g \text{ simple on } \Omega, 0 \leq g \leq f \circ X\right\} \\ &= \sup\left\{\int_\Omega h(X(\omega))dP(\omega); h \text{ simple on } \mathbb{R}, 0 \leq h \circ X \leq f \circ X\right\} \\ &= \sup\left\{\int_\mathbb{R} h(x)p_X(dx); h \text{ simple, } 0 \leq h \leq f\right\} \\ &= \int_\mathbb{R} f(x)dp_X(x). \end{aligned}$$

In the general case write $f = f_+ - f_-$. If f is non-negative but not necessarily bounded, the result still holds but both integrals may be (simultaneously) infinite.

52 By linearity, the quadratic function $g(t) = \mathbb{E}(X^2) + 2t\mathbb{E}(XY) + t^2\mathbb{E}(Y^2) \geq 0$ for all $t \in \mathbb{R}$ and so $4\mathbb{E}(XY)^2 \leq 4\mathbb{E}(X^2)\mathbb{E}(Y^2)$ and the result follows.

If X^2 is integrable put $Y = 1$ in the (just-proved) Cauchy-Schwarz inequality to get $\mathbb{E}(|X|) \leq \mathbb{E}(X^2)^{\frac{1}{2}} < \infty$. So X is integrable and by Problem 30(a), $|\mathbb{E}(X)| \leq \mathbb{E}(|X|)$. Combining the last two inequalities yields $|\mathbb{E}(X)|^2 \leq \mathbb{E}(X^2)$.

53 The first result follows easily from

$$\mathrm{Var}(X) = \mathbb{E}((X-\mu)^2) = \mathbb{E}(X^2) - 2\mu\mathbb{E}(X) + \mu^2 = \mathbb{E}(X^2) - \mu^2.$$

For the second result, replace X by $|X - \mu|$ in the final inequality of Problem 52 to obtain,

$$\mathbb{E}(|X-\mu|)^2 \leq \mathbb{E}((X-\mu)^2) = \mathrm{Var}(X).$$

54 Write $|X| = |X|\mathbf{1}_{\{|X|\leq 1\}} + |X|\mathbf{1}_{\{|X|>1\}}$ and then we get $|X|^m = |X|^m\mathbf{1}_{\{|X|\leq 1\}} + |X|^m\mathbf{1}_{\{|X|>1\}}$. Using the facts that $|x|^m \leq 1$ whenever $|x| \leq 1$ and $|x|^m \leq |x|^n$ whenever $|x| \geq 1$, we find by linearity and monotonicity that

$$\begin{aligned}
\mathbb{E}(|X|^m) &= \mathbb{E}(|X|^m\mathbf{1}_{\{|X|\leq 1\}}) + \mathbb{E}(|X|^m\mathbf{1}_{\{|X|>1\}}) \\
&\leq 1 + \mathbb{E}(|X|^n\mathbf{1}_{\{X>1\}}) \\
&\leq 1 + \mathbb{E}(|X|^n) < \infty.
\end{aligned}$$

55 (X_n) converges to X in probability since given any $\epsilon > 0$ and $c > 0$ we can find $n_0 \in \mathbb{N}$, which we write $n_0 = 2^m + r$ for some natural number m where $r = 0, 1, 2, \ldots, 2^m - 1$, so that $\frac{1}{2^m c} < \epsilon$. Then for all $n > n_0$, by Markov's inequality

$$P(|X_n - X| > c) = P(\mathbf{1}_{A_n} > c) \leq \frac{\mathbb{E}(\mathbf{1}_{A_n})}{c} < \frac{1}{2^m c} < \epsilon.$$

On the other hand (X_n) cannot converge to X almost surely since given any $n \in \mathbb{N}$ no matter how large, we can find $m > n$ so that A_m and A_n are disjoint (with $P(A_n) > 0$) and so $\mathbf{1}_{A_n}(\omega) - \mathbf{1}_{A_m}(\omega) = 1 - 0 = 1$ for all $\omega \in A_n$.

Problems 56–61 are on the part of the course that is non–examinable this year.